One-Dimensional Bubbly Cavitating Flows Through a Converging-Diverging Nozzle

Yi-Chun Wang
Assistant Professor,
Department of Mechanical Engineering,
National Cheng Kung University,
Tainan 701, Taiwan

C. E. Brennen
Professor,
Mail Code: 104-44
Department of Mechanical Engineering
California Institute of Technology,
Pasadena, CA 91125

A nonbarotropic continuum bubbly mixture model is used to study the one-dimensional cavitating flow through a converging-diverging nozzle. The nonlinear dynamics of the cavitation bubbles are modeled by the Rayleigh-Plesset equation. Analytical results show that the bubble/bubble interaction through the hydrodynamics of the surrounding liquid has important effects on this confined flow field. One clear interaction effect is the Bernoulli effect caused by the growing and collapsing bubbles in the nozzle. It is found that the characteristics of the flow change dramatically even when the upstream void fraction is very small. Two different flow regimes are found from the steady state solutions and are termed: quasi-steady and quasi-unsteady. The former is characterized by large spatial fluctuations downstream of the throat which are induced by the pulsations of the cavitation bubbles. The quasi-unsteady solutions correspond to flashing flow. Bifurcation occurs as the flow transitions from one regime to the other. An analytical expression for the critical bubble size at the bifurcation is obtained. Physical reasons for this quasi-static instability are also discussed.

Introduction

One-dimensional bubbly liquid flows in ducts and nozzles represent one of the simplest confined gas-liquid flows. This is an important problem by itself in many engineering applications, but has not, previously, been studied in the context of cavitation bubble/bubble interactions. The nozzle flow is also a useful model of any cavitating flow in which a low pressure region causes the flow to accelerate, for example, the cavitating flow on the suction surface of a hydrofoil. Therefore, study of the one-dimensional accelerating flow with bubble cavitation effects may have value in building up fully nonlinear solutions for practical three-dimensional flows.

In some bubbly flows it is possible to establish a barotropic relation, $p = f(\rho)$, which assumes that the fluid pressure is the function of fluid density only. This implies that all effects caused by bubble content are disregarded except for the compressibility and that the bubbly mixture can be regarded effectively as a single-phase compressible fluid. Tangren et al. (1949) first addressed the barotropic nozzle flow of a two-phase mixture. A summary of this subject can be found, for example, in Brennen (1995). In many practical flows, however, the barotropic criterion is not met. In the present context, the hydrodynamic effects of the flow acceleration cause the bubbles to cavitate and then the flow deceleration makes them collapse. Under these circumstances, the fluid is not barotropic and, as we shall see, the growth and collapse of cavitating bubbles can dramatically change or destabilize the flow.

The flow model used here is a nonlinear continuum bubbly mixture model coupled with the Rayleigh-Plesset equation for the bubble dynamics. This model was first proposed by van Wijngaarden (1968, 1972) and has been used for studying steady and transient shock wave propagation in bubbly liquids without the acceleration of the mean flow (see, for example, Noordzij and van Wijngaarden, 1974; Kameda and Matsumoto, 1995). Ishii et al. (1993) proposed a bubbly flow model and used it to study steady flows through a converging-diverging nozzle. However, by assuming that the gas pressure inside the bubbles is equal to the ambient fluid pressure, they neglected the bubble radial dynamics (as represented by the Rayleigh-Plesset equation) which are dominant mechanisms in a cavitating flow. Morioka and Matsui (1980) and Morioka and Toma (1984) investigated the acoustic dispersion relation for a flowing bubbly liquid using van Wijngaarden's model and Toma and Morioka (1986) examined characteristics of different acoustic modes in flowing bubbly liquid using the same model. Toma et al. (1988) conducted experiments with bubbly liquid flows in a converging-diverging nozzle and recorded the temporal fluctuation characteristics of this kind of flow. However, fully nonlinear solutions of the accelerating bubbly flows with bubble cavitation effects have not, previously, been obtained. The purpose of the present work is to examine what effects bubble dynamics can have on the flow structure.

Basic Equations

Referring to Fig. 1, consider a one-dimensional converging-diverging nozzle with length L and cross-sectional area $A(x)$. The flow direction is in positive x direction and the inlet of the nozzle is located at $x = 0$. The variables in all the figures and equations are non-dimensionalized using the upstream conditions (denoted by subscript s) and the liquid density, ρ_L^*. All quantities with superscript * represent dimensional values. For example, $\eta = \eta^* R_s^{*3}$ is the non-dimensional bubble population per unit liquid volume, where R_s^* is upstream bubble radius.

The continuity and momentum equations of the bubbly flow (references d'Agostino and Brennen, 1983, 1989; Wang, 1996) have the forms

$$\frac{\partial}{\partial t}[(1-\alpha)A] + \frac{\partial}{\partial x}[(1-\alpha)uA] = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{2(1-\alpha)}\frac{\partial C_P}{\partial x} \tag{2}$$

where $\alpha(x, t)$, the bubble void fraction, is related to the bubble radius, $R(x, t)$, by $\alpha(x, t) = 4/3\pi\eta R^3(x, t)/[1 + 4/3\pi\eta R^3(x, t)]$, $u(x, t)$ is the fluid velocity, $C_P(x, t) = (p^*(x, t) -$

Contributed by the Fluids Engineering Division for publication in the JOURNAL OF FLUIDS ENGINEERING. Manuscript received by the Fluids Engineering Division October 28, 1996; revised manuscript received May 6, 1997. Associate Technical Editor: J. Katz.

$p_s^*)/1/2\rho_L^* u_s^{*2}$ is the fluid pressure coefficient, $p^*(x, t)$ is the fluid pressure, p_s^* is the upstream fluid pressure, and u_s^* is the upstream fluid velocity. The liquid has been assumed to be incompressible and the relative motion between the phases has been ignored. Friction between the fluid and the duct wall is also neglected. It is assumed that the upstream bubble population per unit volume of liquid is piecewise uniform and that there is no coalescence or break-up of bubbles in the flow. Since relative motion and the mass of liquid vaporized or condensed are neglected, it follows that η remains both constant and piecewise uniform in the flow. The nondimensional fluid density has been approximated by $\rho \approx (1 - \alpha)$ in (1) and (2) since the liquid density is very much larger than the vapor density. The interactions of the bubbles with the flow are modeled by the Rayleigh-Plesset equation (Knapp et al., 1970; Plesset and Prosperetti, 1977) which connects the local fluid pressure coefficient, C_P, to the bubble radius, R:

$$R\frac{D^2R}{Dt^2} + \frac{3}{2}\left(\frac{DR}{Dt}\right)^2 + \frac{\sigma}{2}(1 - R^{-3k}) + \frac{4}{\mathrm{Re}}\frac{1}{R}\frac{DR}{Dt} + \frac{2}{\mathrm{We}}(R^{-1} - R^{-3k}) + \frac{1}{2}C_P = 0 \quad (3)$$

where $D/Dt = \partial/\partial t + u\partial/\partial x$ is the Lagrangian derivative, $\sigma = (p_s^* - p_V^*)/1/2\rho_L^* u_s^{*2}$ is the cavitation number and p_V^* is the partial pressure of vapor inside the bubble. The partial pressure of noncondensable gas (it is assumed the mass of gas inside each bubble is constant) does not appear explicitly in (3) because the upstream equilibrium condition has been employed to eliminate this quantity. It has also been assumed that the noncondensable gas inside the bubbles behaves polytropically with an index k. If $k = 1$, a constant bubble temperature is implied and $k = \gamma$, the ratio of specific heats of the gas, would model adiabatic behavior. We define a Reynolds number, $\mathrm{Re} = \rho_L^* u_s^* R_s^*/\mu_E^*$ where μ_E^* is the effective viscosity of liquid which incorporates the various bubble damping mechanisms, namely acoustic, thermal, and viscous damping, described by Chapman and Plesset (1971). We also define a Weber number, $\mathrm{We} = \rho_L^* u_s^{*2} R_s^*/S^*$ where S^* is the surface tension of the liquid.

Equations (1), (2), and (3) represent a simple model of one-dimensional flowing bubbly fluid with nonlinear bubble dynamics. Previous investigations have examined the dispersion and stability properties of this model in the linear regimes (see, for example, Biesheuvel and van Wijngaarden, 1984; Morioka and Matsui, 1980; Morioka and Toma, 1984; Toma and Morioka, 1986; Toma et al., 1988). These results helped to identify the propagation modes and the dispersion characteristics of the acoustic waves in a flowing bubbly liquid. However, if the flow is accelerating, simple linearization of the equations of motion is impossible since the mean flow quantities are changing rapidly with both space and time. Analyses of the dynamics of this model then become significantly more complicated and new phenomena may be manifest due to the coupling of flow acceleration and bubble dynamics.

Steady-State Solutions

Only steady flows are considered in the present work. It is assumed that (1), (2), and (3) have steady-state solutions for a constant mass flow rate with upstream conditions denoted by p_s^*, u_s^*, and $\rho_s^* \approx \rho_L^*(1 - \alpha_s) = \rho_L^*/(1 + 4/3\pi\eta^* R_s^{*3})$ where α_s is the upstream void fraction. After dropping all the partial time derivative terms, the governing equations become a system of ordinary differential equations with one independent variable, x:

$$(1 - \alpha)uA = (1 - \alpha_s) = \text{constant} \quad (4)$$

$$u\frac{du}{dx} = -\frac{1}{2(1-\alpha)}\frac{dC_P}{dx} \quad (5)$$

$$R\left(u^2\frac{d^2R}{dx^2} + u\frac{du}{dx}\frac{dR}{dx}\right) + \frac{3u^2}{2}\left(\frac{dR}{dx}\right)^2 + \frac{4}{\mathrm{Re}}\frac{u}{R}\frac{dR}{dx} + \frac{2}{\mathrm{We}}\left(\frac{1}{R} - \frac{1}{R^{3k}}\right) + \frac{\sigma}{2}\left(1 - \frac{1}{R^{3k}}\right) + \frac{1}{2}C_P = 0 \quad (6)$$

The initial or upstream conditions are given by:

$$R(x = 0) = 1, \quad u(x = 0) = 1, \quad C_P(x = 0) = 0 \quad (7)$$

We choose to examine a simple nozzle, $A(x)$, such that

$$A(x) = \begin{cases} \left\{1 - \frac{1}{2}C_{PMIN}\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]\right\}^{-1/2}; & 0 \le x \le L \\ 1; & x < 0 \text{ and } x > L \end{cases} \quad (8)$$

Nomenclature

A = dimensionless cross-sectional area of nozzle, A^*/A_s^*
A^* = cross-sectional area of nozzle
A_s^* = upstream cross-sectional area of nozzle
C_P = fluid pressure coefficient, $(p^* - p_s^*)/1/2\rho_L^* u_s^{*2}$
C_{Pc} = critical pressure coefficient at which flashing occurs
C_{PMIN} = minimum pressure coefficient at throat for pure liquid nozzle flow
L = dimensionless length of the nozzle, L^*/R_s^*
L^* = length of the nozzle
R = dimensionless bubble radius, R^*/R_s^*
R_c = dimensionless critical bubble radius at which flashing occurs
R_s^* = upstream bubble radius
Re = Reynolds number, $\rho_L^* u_s^* R_s^*/\mu_E^*$
S^* = surface tension of the liquid
We = Weber number, $\rho_L^* u_s^{*2} R_s^*/S^*$
k = polytropic index for the gas inside the bubbles
p^* = fluid pressure
p_s^* = upstream pressure
p_V^* = vapor pressure
t = dimensionless time, $t^* u_s^*/R_s^*$
t^* = time
u = dimensionless fluid velocity, u^*/u_s^*
u^* = fluid velocity
u_s^* = upstream fluid velocity
x = dimensionless Eulerian coordinate, x^*/R_s^*
x^* = Eulerian coordinate
α = void fraction of the bubbly fluid
α_b = upstream void fraction at which flashing occurs
α_s = upstream void fraction
η = dimensionless bubble population per unit liquid volume, $\eta^* R_s^{*3}$
η^* = bubble population per unit liquid volume
γ = ratio of specific heats of the gas inside the bubbles
μ_E^* = effective dynamic viscosity of the liquid
ρ = dimensionless fluid density
ρ_L^* = density of the liquid
ρ_s^* = upstream fluid density
σ = cavitation number, $(p_s^* - p_V^*)/1/2\rho_L^* u_s^{*2}$

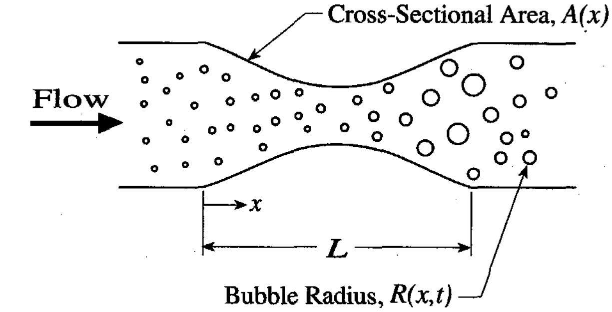

Fig. 1 Notation for bubbly liquid flow in a converging-diverging nozzle

This profile will produce a simple sinusoidal pressure distribution in the case of incompressible flow with the minimum pressure coefficient, C_{PMIN}, located at the nozzle throat, $x = L/2$. The value of $-C_{PMIN}$ relative to the cavitation number, σ, represents the intensity of tension in the flow. If $-C_{PMIN}$ is greater than the cavitation number, σ, the minimum fluid pressure experienced by the individual bubbles will be lower than vapor pressure and the bubbles will cavitate.

Results and Discussion

A fourth-order Runge-Kutta scheme was used to integrate Eqs. (5) and (6). The following flow conditions were chosen to illustrate the computational results. A bubbly fluid, composed of air bubbles ($k = 1.4$) of upstream radius $R_s^* = 100$ μm in water at 20°C ($\rho_L^* = 1000$ kg/m^3, $\mu_L^* = 0.001$ Ns/m^2, $S^* = 0.073$ N/m) flows with $u_s^* = 10$ m/s through a nozzle with profile given by Eq. (8); the nondimensional length of the nozzle is $L = 500$. The minimum pressure coefficient, C_{PMIN}, for the pure liquid flow is chosen as -1. The upstream cavitation number, σ, is set at 0.8, smaller than $-C_{PMIN}$ so that cavitation will occur. The Reynolds number, Re, based on the upstream fluid velocity, the upstream bubble radius, the liquid density, and the effective liquid viscosity is taken as 33. An effective liquid viscosity, $\mu_E^* = 0.03$ Ns/m^2, is used in place of actual liquid viscosity to incorporate the various bubble damping mechanisms (Chapman and Plesset, 1971). Five different upstream void fractions, α_s, of the order of 10^{-6} are used in the computation and the results are shown in Figs. 2 to 5.

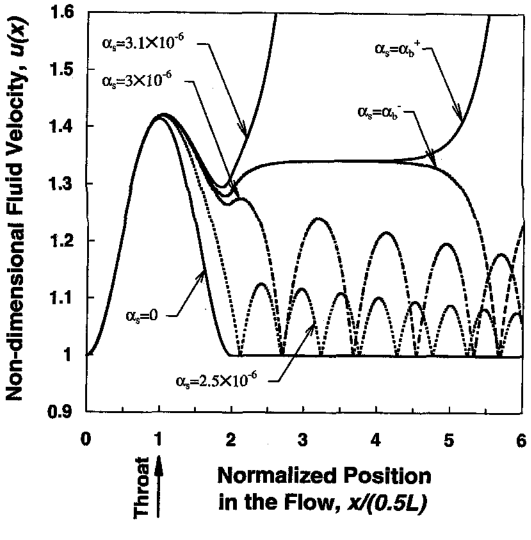

Fig. 2 The nondimensional fluid velocity distribution as a function of the normalized position in the flow for five different upstream void fractions. Labels of $\alpha_s = \alpha_b^-$ and $\alpha_s = \alpha_b^+$ correspond to α_s just below and above the critical value $\alpha_b \approx 3.045 \times 10^{-6}$. The dimensionless length of the nozzle, L, is 500 with the throat located at 0.5L. Other parameters are $\sigma = 0.8$, $C_{PMIN} = -1.0$, Re = 33, and We = 137.

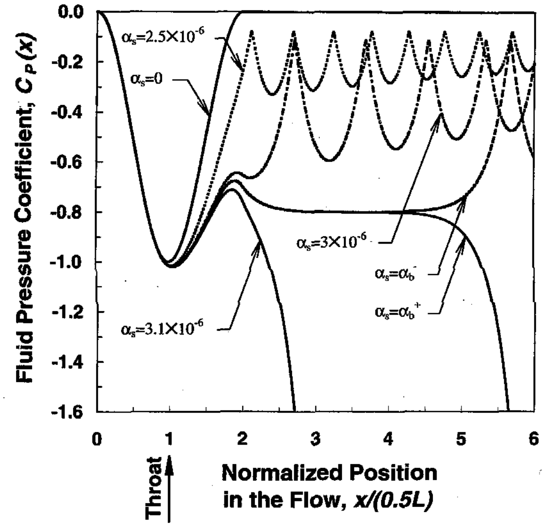

Fig. 3 The fluid pressure coefficient corresponding to Fig. 2

Figure 2 illustrates the fluid velocity. The case of $\alpha_s = 0$ corresponds to the incompressible pure liquid flow. It is notable that even for an upstream void fraction as small as 2.5×10^{-6}, the characteristics of the flow are radically changed from the case without bubbles. Radial pulsation of bubbles results in the downstream fluctuations of the flow. The amplitude of the velocity fluctuation downstream of the nozzle is about ten percent of that of the incompressible flow in this case. As α_s increases further, the amplitude as well as the wavelength of the fluctuations increase. However, the velocity does eventually return to the upstream value due to the bubble damping. In other words, the flow is still "quasi-statically stable." However, as α_s increases to a critical value, α_b (about 3.045×10^{-6} in the present calculation), a bifurcation occurs. The velocity increases dramatically and the flow becomes "quasi-statically unstable." The physical picture of this instability is quite simple: Growth of the cavitation bubbles increases the fluid velocity according to the mass conservation of the flow. The increase of the velocity then causes the fluid pressure to decrease due to the Bernoulli effect. The decrease of the pressure is fed back

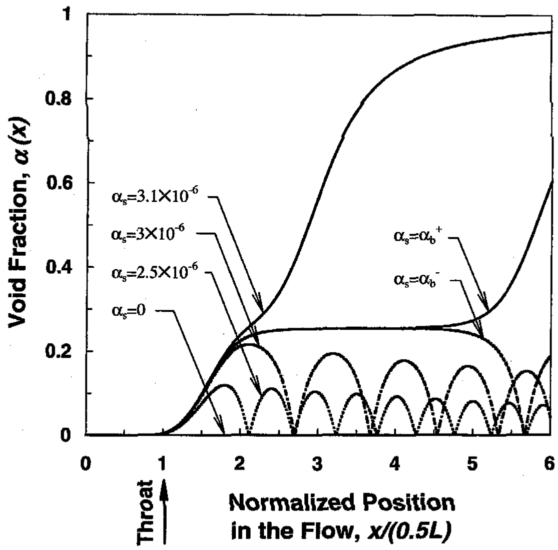

Fig. 4 The void fraction distribution corresponding to Fig. 2

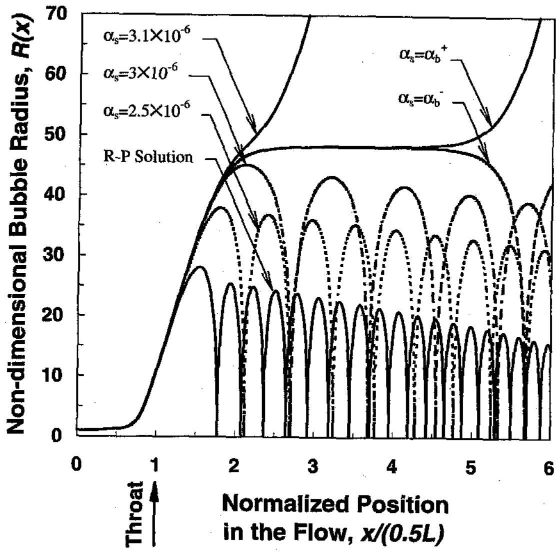

Fig. 5 The nondimensional bubble radius distribution corresponding to Fig. 2. R-P solution represents the solution from the Rayleigh-Plesset equation.

to the Rayleigh-Plesset dynamics and results in more bubble growth.

The corresponding variations in the fluid pressure coefficient are shown in figure 3. In addition to the two different flow regimes, another important feature in the quasi-statically stable flow is the typical frequency associated with the downstream periodicity. This "ringing" will result in acoustic radiation at frequencies corresponding to this wavelength. How this ring frequency relates to the upstream flow conditions remains to be studied. Furthermore, it should be noticed that there is a pressure loss downstream; the fluid pressure does not return to the upstream value except in the case of the pure liquid flow. The only damping mechanism in the present model is due to the bubble damping. Since the viscosity of wall and slip motion between bubbles and liquid are all neglected, the pressure loss is caused by the radial motion of bubbles and represents the "cavitation loss."

Figure 4 illustrates the void fraction distribution in the flow. When the flow becomes quasi-statically unstable, the bubble void fraction, $\alpha(x)$, quickly approaches unity. This means that the flow is flashing to vapor. Moreover we should emphasize that when α becomes large, our model equations, which are limited to flows with small void fraction (for the limitation of void fraction in the present model, see d'Agostino and Brennen, 1989), lose their validity.

Figure 5 indicates the non-dimensional bubble radius distribution in the flow. Due to time lag during the bubble growth phase, bubbles reach the maximum size after passing the nozzle throat. With increase in the upstream void fraction, the maximum size of the bubbles increases and is shifted further downstream. The bubbles grow without bound after reaching the critical radius, R_c, at which flashing begins. Note that R_c is dependent on the cavitation number and the upstream void fraction. An analytical expression for R_c can be found as follows. From figure 5 we note that dR/dx and d^2R/dx^2 both vanish at $R = R_c$. Substitution of these conditions into (6) gives

$$\frac{2}{\mathrm{We}}(R_c^{-1} - R_c^{-3k}) + \frac{\sigma}{2}(1 - R_c^{-3k}) + \frac{1}{2}C_{Pc} = 0 \tag{9}$$

Here C_{Pc} can be found by integrating (4) and (5) by putting $A = 1$ (assuming that the flow exits the nozzle into a length of constant area duct downstream of the nozzle):

$$C_{Pc} = -\frac{24\pi\eta R_c^3}{(3 + 4\pi\eta)^2}\left(1 - \frac{1}{R_c^3}\right) \tag{10}$$

Since $R_c \gg 1$, all the higher order terms ($1/R_c^{3k}$ in (9) and $1/R_c^3$ in (10)) can be neglected. After combining these two equations, one can write

$$R_c^4 - \frac{\sigma}{2\alpha_b(1 - \alpha_b)}R_c - \frac{2}{\alpha_b(1 - \alpha_b)\,\mathrm{We}} = 0 \tag{11}$$

in which $4/3\pi\eta = \alpha_b/(1 - \alpha_b)$ has been used. The third term in (11) can be neglected because, in addition to $R_c \gg 1$, practical values for $2/\mathrm{We}$ are about an order of magnitude less than the values of $\sigma/2$ in the second term. Thus, finally we have:

$$R_c = \left[\frac{\sigma}{2\alpha_b(1 - \alpha_b)}\right]^{1/3} \approx \left[\frac{\sigma}{2\alpha_b}\right]^{1/3} \tag{12}$$

If $R > R_c$, the flow becomes quasi-statically unstable and flashes. In the cases presented here $(\sigma/2\alpha_b)^{1/3} \approx 51$. With known R_c, the expressions for the critical pressure coefficient can be obtained from (10):

$$C_{Pc} = 2\alpha_b(1 - \alpha_b) - \sigma \approx -\sigma \tag{13}$$

Concluding Remarks

Steady cavitating bubbly flows through a converging-diverging nozzle have been examined in the present paper. It was found that the nonlinear bubble dynamics coupled with the equations of motion of the bubbly fluid strongly affect the structure of the flow even for very small bubble populations. Two different flow regimes, distinguished by the parameter $R_c = (\sigma/2\alpha_b)^{1/3}$, (where σ is the cavitation number of the flow and α_b is the upstream void fraction at which the bifurcation occurs) are revealed in the steady state solutions. The flow becomes quasi-statically unstable and flashes to vapor if the radius of the cavitating bubbles is greater than R_c. In this circumstance, the growth of bubbles increases the fluid velocity due to mass conservation of the flow. The velocity increase then causes the fluid pressure to decrease according to the momentum equation. The decrease of the pressure is fed back to the Rayleigh-Plesset equation and results in further bubble growth. In this case the velocity and void fraction of the fluid increase and the pressure coefficient of the flow decreases significantly below the upstream values and the flow flashes to vapor. On the other hand, if the bubbles do not grow beyond R_c, the flow is quasi-statically stable and is characterized by large amplitude spatial fluctuations downstream of the throat.

Finally, we should note that the present work analyzes a simplified internal bubbly flow model with bubble cavitation effects only. Other possible nonequilibrium factors in a real flow, such as thermal nonequilibrium between the phases and nuclei number density distribution in the flow, are excluded. Direct comparison between the present work and previous experimental data are therefore limited. However, the present results show that bubble cavitation may contribute to the void development and downstream oscillation of a bubbly flow in a drastic way.

Acknowledgments

The authors are very grateful for the support for this research provided by National Science Council, Taiwan, R.O.C., under Contract NSC86-2621-E006-039-T and by the Office of Naval Research under Contract N00014-91-J-1295.

References

Biesheuvel, A., and van Wijngaarden, L., 1984, "Two Phase Flow Equations for a Dilute Dispersion of Gas Bubbles in Liquid," *Journal of Fluid Mechanics*, Vol. 148, pp. 301–318.

Brennen, C. E., 1995, *Cavitation and Bubble Dynamics,* Oxford University Press, New York.

Chapman, R. B., and Plesset, M. S., 1971, "Thermal Effects in the Free Oscillation of Gas Bubbles," ASME *Journal of Basic Engineering,* Vol. 93, pp. 373–376.

d'Agostino, L., and Brennen, C. E., 1983, "On the Acoustical Dynamics of Bubble Clouds," ASME *Cavitation and Multiphase Flow Forum,* Houston, TX, pp. 72–75.

d'Agostino, L., and Brennen, C. E., 1989, "Linearized Dynamics of Spherical Bubble Clouds," *Journal of Fluid Mechanics,* Vol. 199, pp. 155–176.

Ishii, R., Umeda, Y., Murata, S., and Shishido, N., 1993, "Bubbly Flows Through a Converging-Diverging Nozzle," *Physics of Fluids A,* Vol. 5(7), pp. 1630–1643.

Kameda, M., and Matsumoto, Y., 1995, "Structure of Shock Waves in a Liquid Containing Gas Bubbles," IUTAM *Symposium on Waves in Liquid/Gas and Liquid/Vapour Two-Phase Systems,* pp. 117–126.

Knapp, R. T., Daily, J. W., and Hammitt, F. G., 1970, *Cavitation,* McGraw Hill, New York.

Morioka, S., and Matsui, G., 1980, "Mechanism and Effect of Dispersion and Dissipation in Nozzle Flows of Bubbly Liquid," *Proceedings of 1st Asian Congress on Fluid Mechanics,* Bangalore, India, A54.

Morioka, S., and Toma, T., 1984, "Stability of Two-Phase Liquid Metal MHD Channel Flows," *Progress in Astronautics & Aeronautics,* Vol. 100, pp. 317.

Noordzij, L., and van Wijngaarden, L., 1974, "Relaxation Effects, Caused by Relative Motion, on Shock Waves in Gas-Bubble/Liquid Mixtures," *Journal of Fluid Mechanics,* Vol. 66, pp. 115–143.

Plesset, M. S., and Prosperetti, A., 1977, "Bubble Dynamics and Cavitation," *Annual Review of Fluid Mechanics,* Vol. 9, pp. 145–185.

Tangren, R. F., Dodge, C. H., and Seifert, H. S., 1949, "Compressibility Effects in Two-Phase Flow," *Journal of Applied Physics,* Vol. 20(7), pp. 637–645.

Toma, T., and Morioka, S., 1986, "Acoustic Waves Forced in Flowing Bubbly Liquid," *Journal of the Physical Society of Japan,* Vol. 55(2), pp. 512–520.

Toma, T., Yoshino, K., and Morioka, S., 1988, "Fluctuation Characteristics of Bubbly Liquid Flow in Converging-Diverging Nozzle," *Fluid Dynamics Research,* Vol. 2, pp. 217–228.

van Wijngaarden, L., 1968, "On the Equations of Motion for Mixtures of Liquid and Gas Bubbles," *Journal of Fluid Mechanics,* Vol. 33, pp. 465–474.

van Wijngaarden, L., 1972, "One-Dimensional Flow of Liquids Containing Small Gas Bubbles," *Annual Review of Fluid Mechanics,* Vol. 4, pp. 369–396.

Wang, Y.-C., 1996, "Shock Waves in Bubbly Cavitating Flows," Ph.D. thesis, California Institute of Technology, Pasadena, California.