ON THE ACOUSTICAL DYNAMICS OF BUBBLE CLOUDS

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1. *INTRODUCTION*

Recently, Mørch [1,2,3,4] Chahine [5,6] and others have focused attention on the dynamics of a cloud or cluster of cavitating bubbles and have expanded on the work of van Wijngaarden [7,8] and others. Unfortunately, there appear to be a number of inconsistencies in this recent work which will require further study before a coherent body of knowledge on the dynamics of clouds of bubbles is established. For example, Mørch and his co-workers [1,2,3] have visualized the collapse of a cloud of cavitating bubbles as involving the inward propagation of a shock wave; it is assumed that the bubbles collapse virtually completely when they encounter the shock. This implies the virtual absence of non-condensable gas in the bubbles and the predominance of vapor. Yet in these circumstances the mixture in the cloud will not have any real sonic speed. As implied by a negative L.H.S. of equation (9), the fluid motion equations for the mixture would be *elliptic* not *hyperbolic* and hence shock wave solutions are inappropriate.

One can visualize several kinds of bubble interaction which could influence the dynamics of a cloud of bubbles. The combined response of the bubbles to an external pressure change will result in volume changes leading to a global accelerating, velocity field. Associated with this velocity field would be pressure gradients which would determine the pressure encountered by individual bubbles within the mixture. It can be shown that such global interactions usually dominate any pressure perturbation experienced by one bubble due to the growth or collapse of a neighbor. The early work of Chahine [5] does not reflect this fact, though in a later paper [6] he does begin to consider the global motion.

We do not intend to resolve these questions in this brief note, but feel it necessary to indicate the lack of established literature on finite clouds of bubbles. In the present note we only wish to delineate the response of a spherical cloud of bubbles to harmonic pressure fluctuations far from the cloud.

2. *BASIC EQUATIONS*

A number of simplifying assumptions are made in order to construct a soluble set of equations which nevertheless model the interactions between the bubbles and the fluid. First, the relative motion between bubble and the surrounding fluid is neglected; the limitations this imposes will be discussed in a later publication. Then it follows that velocity, $\underline{u}$, must satisfy the continuity equation

$$\nabla \cdot \underline{u} = \frac{\eta}{1+\eta\tau}\frac{D\tau}{Dt} \tag{1}$$

where η is the number of bubbles per unit volume of the liquid (assumed incompressible with density, ρ), $\tau(\underline{x},t)$ the volume of an individual bubble and D/Dt is the material derivative, defined as usual by $\partial/\partial t + \underline{u}\cdot\nabla$. Continuity is written in this form rather than in terms of the void fraction, α $(=\eta/(1+\eta\tau))$, for the following reasons. Since relative motion is neglected, the population, η, must satisfy $D\eta/Dt = 0$. Hence the solution of the equations is greatly simplified by considering only those flows in which the population is piecewise constant in a finite number of prescribed material regions. Then, if the values of population η in each of the prescribed material regions is known initially, η remains at those values throughout the motion.

In the equation of motion:

$$\rho\frac{D\underline{u}}{Dt} = -(1+\eta\tau)\nabla p \tag{2}$$

external body forces have been neglected, as have viscous terms. The pressure, $p(\underline{x},t)$, and the velocity, $\underline{u}(\underline{x},t)$, are defined as the corresponding quantities in the absence of local perturbations caused by the growth of any individual bubble. It is interesting to point out that, due to the particular form of the boundary conditions, the Navier-Stokes equations for the flow considered later are satisfied by an irrotational solution. Hence, the only error introduced by the neglect of viscous terms in (2) is related to the small change of the viscosity caused by the presence of the bubbles.

Finally, the bubble volume, $\tau = 4\pi R^3/3$, or, more conveniently, its radius $R(\underline{x},t)$, is determined by the Rayleigh-Plesset equation:

$$\frac{p_B - p}{\rho} = R\frac{D^2R}{Dt^2} + \frac{3}{2}\left(\frac{DR}{Dt}\right)^2 + \frac{2S}{\rho R} \tag{3}$$

where viscous effects on the bubble growth have been neglected. Here S is the surface tension and p_B is the pressure within the bubble, consisting of partial pressures of the vapor, p_v, and non- condensable gas, p_G. The former is assumed constant (neglecting thermal effects) and the non-condensable gas is assumed to consist of a constant mass which behaves with a polytropic index k so that $p_G = p_{G_0}(R_0/R)^{3k}$, where p_{G_0} is the gas partial pressure at a reference radius, R_0. Furthermore, if the reference state is an equilibrium state, then the equilibrium pressure in the liquid is:

$$p_0 = p_{G_0} + p_V - \frac{2S}{R_0} \tag{4}$$

The system of equations (1), (2), and (3) could in theory be solved for $p(\underline{x},t)$, $\underline{u}(\underline{x},t)$ and $\tau(\underline{x},t)$ when supplemented by suitable boundary conditions and the piecewise constant population, η. In practice the non-linearity of the system causes great difficulties in all but the simplest flows.

3. *DYNAMICS OF A BUBBLY CLOUD IN A LIQUID.*

The problem we now address is that of a spherical cloud of bubbles in an unbonded liquid at rest at infinity as shown in Fig. 1. We only consider the case of very low void fraction, so that $\eta\tau << 1$ and the expression $(1+\eta\tau)$ appearing in (1) and (2) can be approximated as unity; then the equations (1) and (2) become:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = \eta\frac{D\tau}{Dt} \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{6}$$

The radius of the cloud is $A(t)$, where $A(0) = A_0$ is known. The size of the bubbles inside the cloud is a function of both

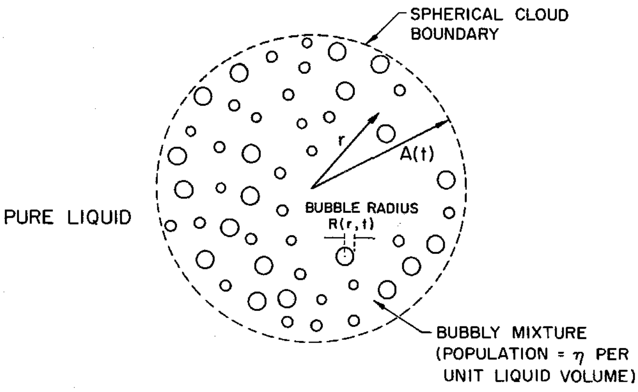

Figure 1. Schematic of spherical cloud of bubbles.

position, r, and time, t, namely $R(r,t)$. For simplicity it is assumed that initially all the bubbles have the same size, $R(r,0) = R_o$. Furthermore we assume that the population per unit liquid volume, η, is uniform and known from the initial conditions. Indeed, if the number of bubbles in the cloud is N, then $\eta = 3N/4\pi(A_o^3 - NR_o^3)$. Outside the cloud where $r \geq A(t)$ there are no bubbles ($\eta = 0$) and the equations reduce to familiar form.

We examine first the linearized form of these equations which will simultaneously provide the response of the cloud to small oscillations in the pressure at infinity, $p_\infty(t)$, and the initial motion of a cloud in equilibrium when subject to a step function change in p_∞. Then:

$$R(r,t) = R_o[1 + \varphi(r,t)] \ ; \ r < A(t) \tag{7}$$

where $\varphi \ll 1$. Under these circumstances it is readily shown that, since the velocity u is of the order of φ (equation (5)), then the convective component of the material derivative is of order φ^2 and D/Dt can be replaced by $\partial/\partial t$ if only terms of order φ are to be considered. Then it follows from the Rayleigh equation (3) that to order, φ,

$$p(r,t) = p_o - \rho R_o^2 \left[\frac{\partial^2 \varphi}{\partial t^2} + \omega_B^2 \varphi\right] \ ; \ r < A(t) \tag{8}$$

where:

$$\omega_B^2 = 3k\,\frac{p_{G_o}}{\rho R_o^2} - \frac{2S}{\rho R_o^3} \tag{9}$$

and ω_B is the natural frequency of oscillation of a single bubble in an infinite liquid [7,9].

If it is assumed that the bubbles are in stable equilibrium in the initial or mean state (p_{G_o}, R_o) so that $3kp_{G_o} > 2S/R_o$ [9,10], then ω_B is real. Finally, upon substitution of (7) and (8) into (5) and (6) and elimination of $u(r,t)$, one obtains the following equation for the function $\varphi(r,t)$ in the domain $r < A(t)$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left[\frac{\partial^2 \varphi}{\partial t^2} + \omega_B^2 \varphi\right]\right] - 4\pi\eta R_o\,\frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{10}$$

The incompressible single phase flow outside the bubble cloud ($r \geq A(t)$) must have the standard classical solution of the form:

$$u(r,t) = \frac{C(t)}{r^2} \ ; \ r \geq A(t) \tag{11}$$

$$p(r,t) = p_\infty(t) + \frac{\rho}{r^2}\frac{dC(t)}{dt} + O(C^2(t)) \ ; \ r \geq A(t) \tag{12}$$

where $C(t)$ is of perturbation order. It follows that, to the first order in $\varphi(r,t)$, the continuity of $u(r,t)$ and $p(r,t)$ at the interface between the cloud and the pure liquid results in the following boundary condition for $\varphi(r,t)$:

$$\left(1 + A_o \frac{\partial}{\partial r}\right)\left[\frac{\partial^2 \varphi}{\partial t^2} + \omega_B^2 \varphi\right]_{r=A_o} = \frac{p_o - p_\infty(t)}{R_o^2 \rho} \tag{13}$$

The linearized solution of equation (10) for small periodic oscillations of frequency ω of the far field pressure in the liquid $p_\infty(t) = p_o + Re\{\tilde{p}_\infty e^{i\omega t}\}$ takes the form:

$$\varphi(r,t) = -\frac{1}{\rho R_o^2} Re\left\{\frac{\tilde{p}_\infty}{\omega_B^2 - \omega^2}\,\frac{e^{i\omega t}}{\cos\lambda A_o}\,\frac{\sin\lambda r}{\lambda r}\right\} \ ; \ r < A_o \tag{14}$$

where:

$$\lambda^2 = 4\pi\eta R_o\,\frac{\omega^2}{\omega_B^2 - \omega^2} \tag{15}$$

Another possible solution involving $(\cos\lambda r)/\lambda r$ has been eliminated since $\varphi(r,t)$ must clearly be finite as $r \to 0$. Therefore in the domain $r < A_o$:

$$R(r,t) = R_o - \frac{1}{\rho R_o} Re\left\{\frac{\tilde{p}_\infty}{\omega_B^2 - \omega^2}\,\frac{e^{i\omega t}}{\cos\lambda A_o}\,\frac{\sin\lambda r}{\lambda r}\right\} \tag{16}$$

then

$$u(r,t) = \frac{1}{\rho} Re\left\{i\,\frac{\tilde{p}_\infty}{\omega}\,\frac{1}{r}\left[\frac{\sin\lambda r}{\lambda r} - \cos\lambda r\right]\frac{e^{i\omega t}}{\cos\lambda A_o}\right\} \tag{17}$$

$$p(r,t) = p_o - Re\left\{\tilde{p}_\infty \frac{\sin\lambda r}{\lambda r}\,\frac{e^{i\omega t}}{\cos\lambda A_o}\right\} \tag{18}$$

The entire flow has therefore been determined in terms of the prescribed quantities A_o, R_o, η, ω and p_∞.

4. *RESULTS*

We examine first the natural modes and frequencies of oscillation of the cloud. From (14) note that if $\tilde{p}_\infty$ were zero, oscillations only occur if:

$$\omega = \omega_B \ \ or \ \ \lambda_n A_o = (2n-1)\frac{\pi}{2} \ , \ n = 0, \pm 1, \pm 2 \ \cdots \tag{19}$$

it follows from the expression (15) for λ that the natural frequencies, ω_n, of the cloud are:

(i) $\omega_\infty = \omega_B$, namely the natural frequency of an individual bubble in an infinite liquid, and

$$(ii) \ \ \omega_n = \omega_B\left[1 + \frac{16\eta R_o A_o^2}{\pi(2n-1)^2}\right]^{-1/2} \ ; \ n = 1,2, \ \cdots \tag{20}$$

which is an infinite series of frequencies of which ω_1 is the lowest. The higher frequencies approach ω_B as u tends to infinity.

The lowest natural frequency, ω_1, can be written in terms of the initial void fraction $\alpha_o = \eta\tau_o/((1+\eta\tau_o)$ (which must be much less than unity for the validity of the analysis) as

$$\omega_1 = \omega_B\left[1 + \frac{4}{3\pi^2}\left(\frac{A_o}{R_o}\right)^2 \frac{\alpha_o}{1-\alpha_o}\right]^{-1/2} \tag{21}$$

Hence, the natural frequencies of the cloud will extend to frequencies much smaller than the individual bubble frequency, ω_B, if the initial void fraction, α_o, is much larger than the square of the ratio of bubble size to cloud size ($\alpha_0 \gg R_0^2/A_0^2$). If the reverse is the case ($\alpha_0 \ll R_0^2/A_0^2$), all the natural frequencies of the cloud are contained in a small range just below ω_B.

Typical natural modes of oscillation of the cloud are depicted in Fig. 2, where normalized amplitudes of the bubble radius and pressure fluctuations are shown as functions of position, r/A_0, within the cloud.

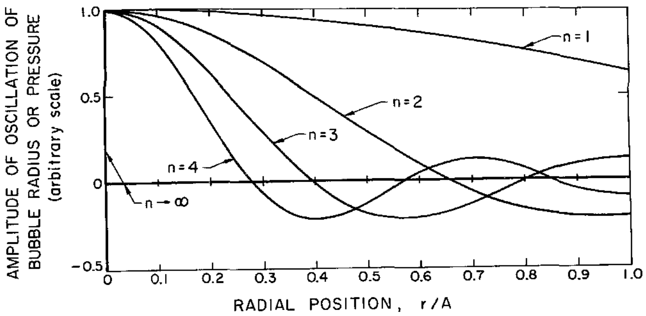

Figure 2. Natural mode shapes as a function of radial position within the bubble cloud. The arbitrary vertical scale represents the amplitude of the bubble radius oscillations and the pressure oscillations for modes n = 1,2,3,4 and $n \to \infty$. The slopes of these waves are proportional to the radial velocity oscillations.

The amplitude of the radial velocity oscillations is proportional to the slope of these curves. Since each bubble is supposed to react to a uniform far field pressure, the validity of the model is limited to wave numbers, n, such that $n \ll A_0/R_0$. Note that the first mode involves almost uniform oscillations of the bubbles at all radial positions within the cloud. Higher modes involve amplitudes of oscillation near the center of the cloud which become larger and larger relative to the amplitudes in the rest of the cloud. In effect, an outer shell of bubbles essentially shields the exterior fluid from the oscillations of the bubbles in the central core, with the result that the pressure oscillations in the exterior fluid are of smaller amplitude for the higher modes.

Fig. 3 shows the forced response amplitude of the bubbles at the center (r=0) and surface of the cloud (r=A_0) as a function of frequency, ω, for a typical case of $\alpha_0 A_0^2/R_0^2$=10. Note the decline in the response at the surface relative to that in the center of the cloud for frequencies less than ω_B. Also note the nodes in the surface response at frequencies between the natural frequencies. The response of the cloud to frequencies greater than ω_B is quite different as illustrated in Fig. 4. Note that, while the entire cloud responses in a fairly uniform manner for $\omega<\omega_B$, only a surface layer of bubbles exhibits significant response when $\omega>\omega_B$. In the latter case the entire core of the cloud is essentially shielded by this outer layer.

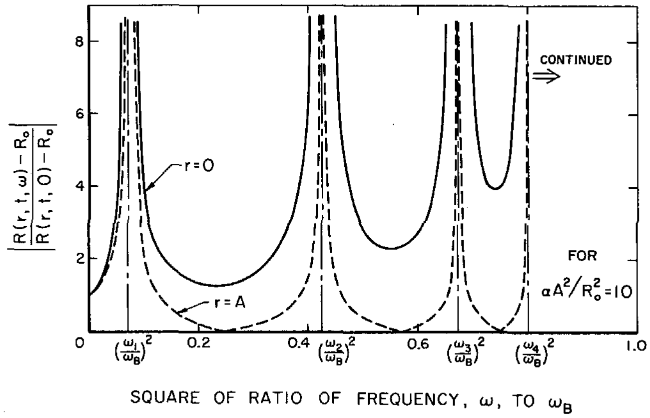

Figure 3. Forced response of bubble cloud as a function of $(\omega/\omega_B)^2$. *Amplitudes of the normalized bubble radius oscillations at the center* (r = 0) and surface (r = A_0) of the cloud are shown for the case of $\alpha_0 A_0^2/R_0^2$ = 10.

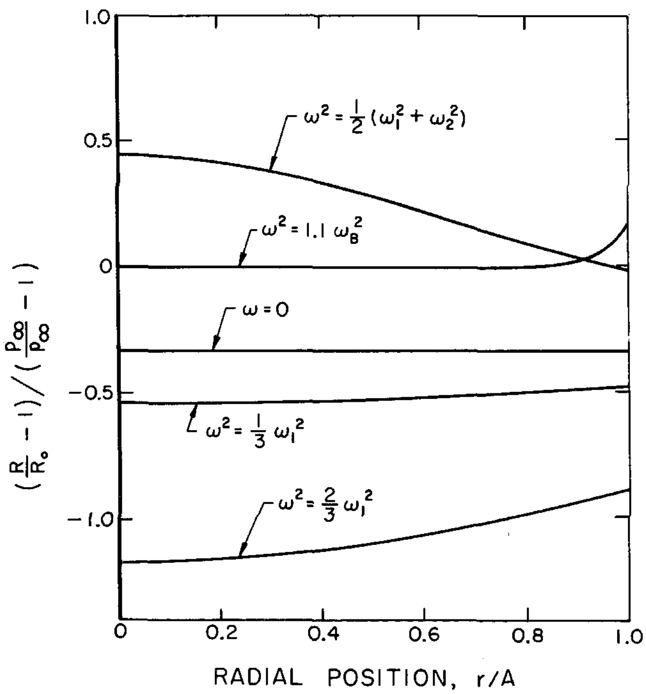

Figure 4. Forced response of the bubble radius oscillations as a function of radial position within the cloud for various frequencies, ω, as indicated (for the case $\alpha_0 A_0^2/R_0^2$=10).

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