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On the flow in an annulus surrounding a whirling cylinder

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When fluid in an annulus between two cylinders is set in motion by whirling movements of one or both of the cylinders, dynamic forces are imposed by the fluid on the cylinders. Knowledge of these forces is frequently important, indeed often critical, to the engineer designing rotor systems or journal bearings. Quite general solutions of the Navier–Stokes equations are presented for this problem and are limited only by restrictions on the amplitude of the whirl motion. From these solutions, the forces are derived under a wide variety of circumstances, including large and small annular widths, high and low Reynolds numbers and with and without a mean flow created by additional net rotation of one or both of the cylinders.

1. Introduction

The flow in an annulus between rotating cylinders must rank high on any list of the most extensively studied flows. Viscometric and hydrodynamic stability aspects of the flow between coaxial cylinders have been studied in depth. Lubrication analyses based on the generalized Reynolds equation have been widely used for the study of journal bearings where the annular gap is small. Thus the engineer faced with a design problem involving journal bearings or rotor shafts surrounded by a fluid annulus can make accurate predictions of the nature of the fluid flow and the steady-state forces involved in his design. But often knowledge of the dynamic stability of the system is of comparable importance; accurate prediction, for example, of the whirling speeds of the inner cylinder or rotor is often a primary concern. Considering the wealth of material on annular flow, it is somewhat surprising to find a comparative paucity of information on the nature of the flow and forces in an annulus between coaxial cylinders when either or both are performing whirling motions; that is to say the position of the cylinder axes in a cross-sectional plane moves in a circle around some mean position or axis as indicated in figure 1. It is clear that accurate prediction of the structural dynamics not only of the inner cylinder (termed the 'rotor' for convenience, even though in one problem studied it does not actually rotate) but also of the outer cylinder or 'stator' requires knowledge of the unsteady forces imposed by the fluid on these components owing to the whirling motion. In general these forces will comprise a force in the instantaneous direction of whirl deflexion OC, which is often

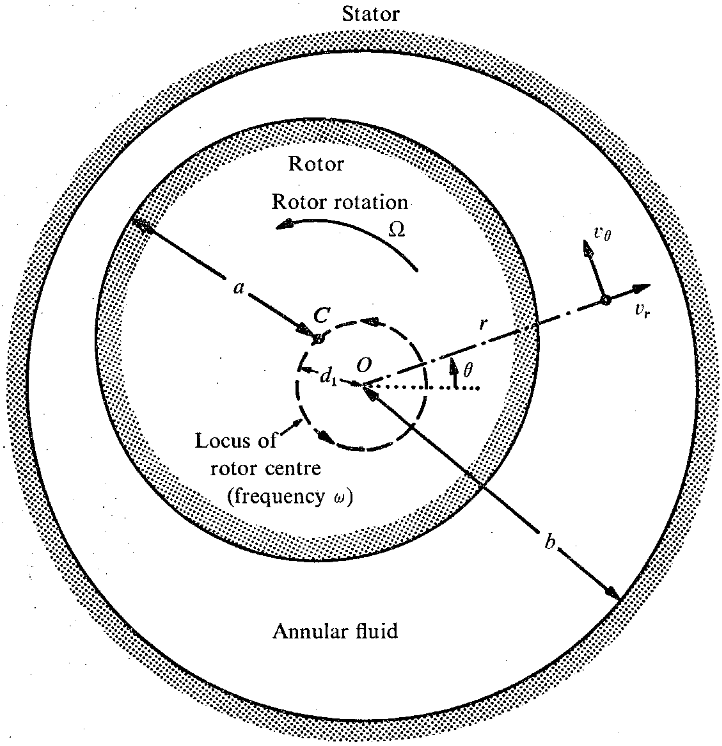

FIGURE 1. Schematic diagram illustrating rotor whirl and general rotation. For simplicity stator whirl motion is not shown in the figure.

equivalent to an 'added mass', and a viscous drag or lubrication force normal to that direction.

One area in which the dynamic aspects of these flows has been extensively studied is in the field of journal-bearing lubrication (e.g. Sternlicht 1965). Almost exclusively such analyses are based on the generalized Reynolds equation (e.g. Tipei 1962) and are therefore restricted to situations in which the annular gap width is very small. Furthermore the Reynolds number is generally small and fluid inertial effects are usually neglected. Recently, however, there has been some interest in inertial effects; Milne (1965), for example, used modifications of the Reynolds equation in order to evaluate second-order inertial effects at low Reynolds numbers. Fritz (1970) has also used lubrication equations in an attempt to estimate dynamic forces at high Reynolds numbers. But it appears that the dynamics of the flow for larger gap widths and the validity of the lubrication theories in the various asymptotic limits of high and low Reynolds numbers and small gap widths are largely unknown.

The particular technological problem which originally motivated this study is worth describing. Designs for liquid-sodium pumps in the cooling system of proposed nuclear generating plants have the pump buried in the core while its prime mover is outside of the radioactive shielding. In some candidate designs the long rotor connecting them is housed in a cylindrical casing; this is, in turn, surrounded by the pump housing and liquid sodium fills both annuli. Thus, if the rotor performs whirling motions (called for convenience 'whirl with rotation') forces transmitted by the fluid in the inner annulus would cause the intermediate cylinder or casing to deflect structurally and thus perform whirl motions *without* any net rotation. The problem in the inner annulus is thus one of 'whirl with rotation' while that in the outer annulus is one of 'whirl without rotation'. The forces transmitted in both annuli must be known before the basic rotor dynamics can be analysed since the fluid force on the rotor is a function of the whirl deflexion of the intermediate casing, which is in turn a function of the forces imparted by the fluid motions in the outer annulus. Furthermore neither annulus is small compared with the radial dimensions and hence a lubrication theory would be of dubious validity.

The objective of the present paper is to evaluate the fluid motions and forces for a variety of situations by starting with flow solutions of the Navier–Stokes equations. Though fluid motion in the axial direction can be a significant feature, particularly in journal-bearing lubrication (e.g. Sternlicht 1965), it will be assumed zero throughout this paper; the extension of the present analysis to include axial velocities is very complex algebraically. Furthermore the solutions are all restricted to whirl amplitudes which are very small compared with the other dimensions; such is the case in the problem mentioned above, where amplitudes of thousandths of an inch are of interest for radial dimensions of the order of inches or feet.

The frequency of the whirl motion is denoted by ω. In addition the cylinders may also be rotating about their own axes. For the most part we shall treat in detail only the common situation in which the rotor has some mean angular velocity Ω, where in general $\Omega \neq \omega$. The case of whirl without rotation is thus $\Omega = 0$ while that of conventional or synchronous whirl with rotation is $\Omega = \omega$. There do however exist other forms of whirl with rotation such as half-frequency whirl for which $\Omega \neq \omega$. Furthermore conventional steady-state operation of a journal bearing also fits into this parametric system when $\omega = 0$, $\Omega \neq 0$. Thus we perform a general analysis for independent ω and Ω. The solutions are, however, formulated only for whirl amplitudes and frequencies which are independent of time. In this sense they are analogous to the steady-whirl solutions of lubrication theory (Sternlicht 1965).

As indicated in figure 1, the notation used throughout will be as follows: a and b refer to the radii of the inner and outer cylinders and ρ, μ and ν are respectively the density and dynamic and kinematic viscosities of the liquid, which is assumed incompressible. Polar co-ordinates (r, θ) with origin at the position of the mean axis are used; $z = re^{i\theta}$ denotes vector position, t is time and (v_r, v_θ) are the fluid velocities in the (r, θ) directions. The position of the axis of the inner cylinder is then denoted by z_{c1}, where

$$z_{c1} = d_1 e^{i\omega t}, \quad d_1 \ll a, b, \tag{1}$$

where d_1 is the whirl amplitude. For generality we also describe the deflexion of the outer cylinder (if any) by the position of its axis

$$z_{c2} = d_2 e^{i(\omega t+\phi)}, \quad d_2 \ll a, b, \tag{2}$$

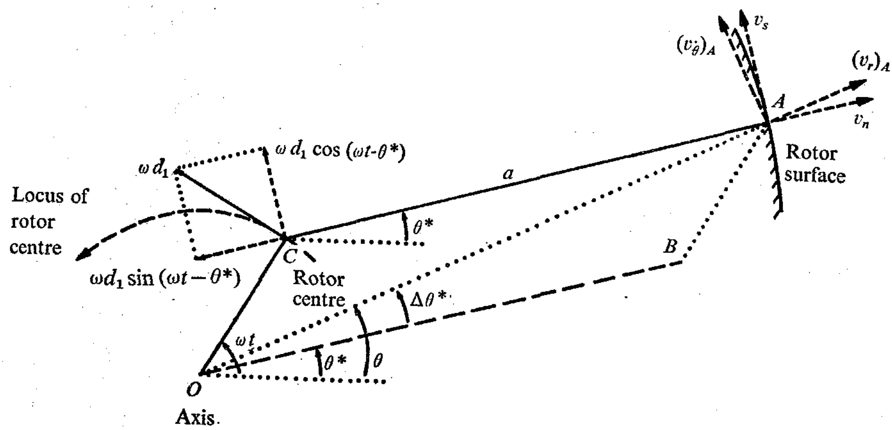

FIGURE 2. Diagram illustrating the component motions of the rotor surface and the derivation of the boundary conditions.

where ϕ is some phase angle, determined from structural as well as hydrodynamic considerations. A whirl amplitude d_2 could arise, for example, from the structural response of the outer cylinder to the forces transmitted through the fluid.

The fluid motions in the annulus are divided into steady components (if any), denoted by (v_r^0, v_θ^0), and oscillatory components (v_r^1, v_θ^1), which are induced by the whirl.

2. Boundary conditions

It is convenient to develop general boundary conditions relating the fluid velocities $(v_r, v_b)_{r=a,b}$ to the motions of the inner and outer cylinder. It will be assumed that the *mean* or time-averaged flow in the annulus is of the form

$$v_r^0 = 0, \quad v_\theta^0 = V(r), \tag{3}$$

which is in accord with zero normal velocity on the cylinder surfaces:

$$(v_r^0)_a = 0, \quad (v_r^0)_b = 0. \tag{4}$$

For convenience, the azimuthal fluid velocities at the inner and outer cylinder surfaces will be denoted by V_a and V_b respectively, so that $V(a) \equiv V_a$ and $V(b) \equiv V_b$.

The above may be envisaged as the $O(1)$ terms in a series solution in ascending powers of d_1 and d_2; then the boundary conditions on the oscillatory velocities are clearly $O(d_1, d_2)$. The boundary condition on the rotor is developed with the aid of figure 2. The velocity of a general material point A on the surface of the rotor is given by superposition of the rotational velocity Ωa about the rotor centre C and the instantaneous velocity of that centre from the relation (1). Thus the velocities normal and tangential to the *instantaneous surface* at A are v_n and v_s, where

$$(v_n)_A = \omega d_1 \sin(\theta^* - \omega t), \tag{5}$$

$$(v_s)_A = \Omega a + \omega d_1 \cos(\theta^* - \omega t). \tag{6}$$

In a viscous flow solution the no-slip conditions require that the fluid velocities at A are identical with (5) and (6). On the other hand, an inviscid analysis requires only (5) and $(v_s)_A$ must be considered arbitrary. We shall attempt to treat both cases simultaneously.

The angle between the polar line OA and the radius CA is $\Delta\theta^*$, where $a\Delta\theta^* = -d_1 \sin(\theta^* - \omega t) + O(d_1^2)$. It follows from (5) and (6) that the radial and azimuthal fluid velocities at A are

$$(v_r)_A = (\omega - (v_s)_A)\, d_1 \sin(\theta^* - \omega t) + O(d_1^2), \tag{7}$$

$$(v_\theta)_A = \Omega a + \omega d_1 \cos(\theta^* - \omega t) + O(d_1^2). \tag{8}$$

In the viscous solution $(v_s)_A$ in (7) can be replaced by Ωa; in the inviscid case only the first relation applies and $(v_s)_A$ must be set equal to the mean tangential fluid velocity V_a. But the point A has polar co-ordinates $r = a + d_1 \cos(\theta^* - \omega t)$, $\theta = \theta^* - d_1 \sin(\theta^* - \omega t)/a$ and thus by a Taylor series expansion

$$(v_r)_A = (v_r)_{a,\theta} + d_1 \cos(\theta - \omega t)\left(\frac{\partial v_r}{\partial r}\right)_{a,\theta} - \frac{d_1}{a}\sin(\theta - \omega t)\left(\frac{\partial v_r}{\partial \theta}\right)_{a,\theta} + O(d_1^2), \tag{9}$$

$$(v_\theta)_A = (v_\theta)_{a,\theta} + d_1 \cos(\theta - \omega t)\left(\frac{\partial v_\theta}{\partial r}\right)_{a,\theta} - \frac{d_1}{a}\sin(\theta - \omega t)\left(\frac{\partial v_\theta}{\partial \theta}\right)_{a,\theta} + O(d_1^2). \tag{10}$$

Such an expansion is necessary because at least one of the derivatives is $O(1)$ in general, namely $\partial v_\theta/\partial r$. The boundary conditions are then obtained in a convenient form (i.e. all quantities evaluated at $r = a$) by eliminating $(v_r)_A$ and $(v_\theta)_A$ among (7)–(10) and employing the mean flow (3) to evaluate the $O(d_1)$ terms arising from the derivatives. The $O(1)$ terms merely confirm the mean flow boundary conditions. Neglecting terms $O(d_1^2)$ results in the following boundary conditions on the oscillatory velocities:

$$(v_r^1)_{r=a} = \left\{\omega - \frac{V_a}{a}\right\} d_1 \sin(\theta - \omega t) = \mathrm{Re}\left[-i\left(\omega - \frac{V_a}{a}\right) d_1 e^{i(\theta - \omega t)}\right], \tag{11}$$

$$(v_\theta^1)_{r=a} = \left\{\omega - \left(\frac{\partial V}{\partial r}\right)_{r=a}\right\} d_1 \cos(\theta - \omega t) = \mathrm{Re}\left[\left\{\omega - \left(\frac{\partial V}{\partial r}\right)_{r=a}\right\} d_1 e^{i(\theta - \omega t)}\right]. \tag{12}$$

In the inviscid solution only the first is applicable and V_a is not necessarily equal to Ωa; in the viscous solution both must be used and $V_a = \Omega a$. Boundary conditions on the stator are obtained in precisely the same way and analogous comments apply:

$$(v_r^1)_{r=b} = \mathrm{Re}\left[-i(\omega - V_b/b)\, d_2 e^{i(\theta - \omega t - \phi)}\right], \tag{13}$$

$$(v_\theta^1)_{r=b} = \mathrm{Re}\left[\{\omega - (\partial V/\partial r)_{r=b}\}\, d_2 e^{i(\theta - \omega t - \phi)}\right]. \tag{14}$$

3. Equations of fluid motion

We shall now lay the general foundations for the particular fluid-mechanical solutions of later sections. Assuming an incompressible Newtonian liquid and defining a stream function ψ and a vorticity ζ in the conventional manner such that

$$v_r = r^{-1}\,\partial\psi/\partial\theta, \quad v_\theta = -\,\partial\psi/\partial r, \quad \zeta = -\nabla^2\psi, \tag{15}$$

the Navier–Stokes equations require that

$$\nu\nabla^2\zeta = \frac{\partial\zeta}{\partial t}+v_r\frac{\partial\zeta}{\partial r}+\frac{v_\theta}{r}\frac{\partial\zeta}{\partial\theta}, \tag{16}$$

where ν is the kinematic viscosity of the liquid. The solution for a mean flow $(v_r^0, v_\theta^0, \zeta^0)$ of the form (3) is simply

$$v_r^0 = 0, \quad \zeta^0 = (bV_b-aV_a)/(b^2-a^2), \tag{17}$$

$$v_\theta^0 = V(r) = \frac{ab(bV_a-aV_b)}{(b^2-a^2)}\frac{1}{r}+\frac{bV_b-aV_a}{b^2-a^2}r = \frac{E}{r}+Fr, \quad \text{say}. \tag{18}$$

Of course, for the case of swirl without rotation there is no such mean flow and $V_a = V_b = 0$. For stable laminar flow in the case where the rotor speed is Ω and the stator does not rotate $V_a = a\Omega$ and $V_b = 0$. At high Reynolds numbers, where in the mean the flow may consist of a core flow bounded by thin boundary layers on both cylinder surfaces, V_a and V_b may be envisaged as the mean fluid velocities at the boundaries of the core. In a purely inviscid flow, of course, the term $O(r)$ in (18) must be zero and $bV_b = aV_a$. Now when the $O(d^2)$ terms are neglected the equation governing the oscillatory vorticity ζ^1 is

$$\frac{\partial\zeta^1}{\partial t}+\frac{v_\theta^0}{r}\frac{\partial\zeta^1}{\partial\theta} = \nu\nabla^2\zeta^1, \tag{19}$$

where v_θ^0 is given by (18).

4. General solution

Define a modified radial variable η and a complex constant β as

$$\eta = [i(\omega-F)/\nu]^{\frac{1}{2}}r, \quad \beta = [1+iE/\nu]^{\frac{1}{2}}, \tag{20}$$

where the roots with a positive real part are implied in both cases. Then the appropriate general solution of (19) is

$$\zeta^1 = \operatorname{Re}\,[\{i(\omega-F)/\nu\}^{\frac{1}{2}}\, Z_\beta(\eta)\, e^{i(\theta-\omega t)}], \tag{21}$$

where, as is common practice, $Z_\beta(\eta)$ denotes the linear combination of Bessel functions $AJ_\beta(\eta)+BY_\beta(\eta)$, where A and B are arbitrary constants to be determined. It is interesting to note that the potential-vortex component of the mean annular flow affects only the argument of the Bessel functions while the solid-body-rotation component affects only the order of the Bessel functions.

When (21) is integrated the general solution for the velocity components can be written as

$$v_r^1 = \operatorname{Re}\left[i\left\{\frac{C}{r^2}+D-\frac{1}{2}\int_{\eta_a}^{\eta} Z_\beta(\eta)\,d\eta+\frac{1}{2\eta^2}\int_{\eta_a}^{\eta}\eta^2 Z_\beta(\eta)\,d\eta\right\}e^{i(\theta-\omega t)}\right], \tag{22}$$

$$v_\theta^1 = \operatorname{Re}\left[\left\{\frac{C}{r^2}-D+\frac{1}{2}\int_{\eta_a}^{\eta} Z_\beta(\eta)\,d\eta+\frac{1}{2\eta^2}\int_{\eta_a}^{\eta}\eta^2 Z_\beta(\eta)\,d\eta\right\}e^{i(\theta-\omega t)}\right], \tag{23}$$

where η_a, η_b are respectively the values of η at $r = a, b$. These contain four constants, which must be determined from the boundary conditions by comparing (22) and (23) with (11)–(14). Thus

$$A = \frac{2(\omega-F)(d_1-d_2e^{-i\phi})}{S}\left[\frac{iE}{\nu}\int_{\eta_a}^{\eta_b} Y_\beta(\eta)\,d\eta - \int_{\eta_a}^{\eta_b}\eta^2 Y_\beta(\eta)\,d\eta\right], \tag{24}$$

$$B = -\frac{2(\omega-F)(d_1-d_2e^{-i\phi})}{S}\left[\frac{iE}{\nu}\int_{\eta_a}^{\eta_b} J_\beta(\eta)\,d\eta - \int_{\eta_a}^{\eta_b}\eta^2 J_\beta(\eta)\,d\eta\right], \tag{25}$$

where η_a, η_b are the values of η at $r = a$, b respectively and

$$S = \int_{\eta_a}^{\eta_b} J_\beta(\eta)\,d\eta\int_{\eta_a}^{\eta_b}\eta^2 Y_\beta(\eta)\,d\eta - \int_{\eta_a}^{\eta_b} Y_\beta(\eta)\,d\eta\int_{\eta_a}^{\eta_b}\eta^2 J_\beta(\eta)\,d\eta. \tag{26}$$

Also

$$C = Ed_1, \quad D = -(\omega-F)\,d_1. \tag{27}$$

It is instructive, indeed necessary, to pause here and consider how this general solution behaves at large Reynolds numbers assuming, for the moment, that the mean flow remains stable and laminar. Using the generalized Debye asymptotic values for Bessel functions in which both the order and argument tend to large values one finds by evaluating A, B and S and substituting into (22) that for large β

$$v_r^1 \to \mathrm{Re}\,[i\{C^*/r^2+D^*\}\,e^{i(\theta-\omega t)}], \tag{28}$$

where

$$C^* = Ed_1 - \frac{a^2b^2}{(b^2-a^2)}\left\{\omega - F - \frac{E}{b^2}\right\}(d_1-d_2\,e^{-i\phi}), \tag{29}$$

$$D^* = -(\omega-F)\,d_1 + \frac{b^2}{(b^2-a^2)}\left\{\omega - F - \frac{E}{b^2}\right\}(d_1-d_2e^{-i\phi}), \tag{30}$$

and this is precisely the inviscid result that would be obtained if the radial velocity conditions (11) and (13) were alone applied to a general inviscid solution of the form (22) and (23) with $A = B = 0$. On the other hand the same limiting process applied to v_θ^1 yields

$$v_\theta^1 = \lim_{|\beta|\to\infty}\mathrm{Re}\left[\left\{\frac{C^*}{r^2} - D^* + \frac{2(d_1-d_2\,e^{-i\phi})}{(b^2-a^2)}H(r)\right\}e^{i(\theta-\omega t)}\right], \tag{31}$$

where

$$H(r) = \{(\omega-F)a^2-E\}\left(\frac{r}{b}\right)^{1+\beta} - \{(\omega-F)b^2-E\}\left(\frac{r}{a}\right)^{1-\beta}. \tag{32}$$

This is identical to the inviscid result mentioned above *except* for the term involving $H(r)$. In the interior of the flow ($a < r < b$) it is clear that $H(r)\to 0$ when the Reynolds number becomes large and hence $|\beta| \to +\infty$. But when $r = a$ *or* $r = b$, i.e. exactly on the boundaries, the proper asymptotic limit is

$$H \to \begin{cases}\{(\omega-F)a^2-E\} & \text{for}\quad r = b, \quad |\beta| \to +\infty, & (33)\\ \{(\omega-F)b^2-E\} & \text{for}\quad r = a, \quad |\beta| \to +\infty, & (34)\end{cases}$$

and with these values for H the azimuthal velocity conditions (12) and (14) are satisfied exactly (as well as the radial velocity conditions). On the other hand the azimuthal velocity conditions are *not* satisfied by the inviscid solution where one takes $H = 0$ throughout and then finds v_θ^1 at $r = a, b$. It must be clear that the physical interpretation of these results is that a boundary layer forms on each surface with a predominantly inviscid perturbation flow in the core between the boundary layers. The inviscid solution with $H = 0$ is the velocity distribution in the *core* and the values for v_θ^1 obtained with $H = 0$ at $r = a, b$ approximate the velocities on the core/boundary-layer interfaces. On the other hand the $H(r)$ contributions to v_θ^1 represent the velocity 'defects' within the boundary layers and the values of these defects are such that the no-slip conditions are satisfied at the material surfaces. Representative thicknesses δ_a and δ_b for the boundary layers are readily obtained by noting the points in the flow $r = a+\delta_a$ and $r = b-\delta_b$, at which the defects are one-half of the defects at the walls, i.e.

$$\delta_a = a[2^{(\nu/E)^{\frac{1}{2}}}-1], \quad \delta_b = b[1-2^{-(\nu/E)^{\frac{1}{2}}}]. \tag{35}$$

There is however a difficulty in this limit process. Note that the solution for large $|\beta|$ involves asymptotically large values of $\partial v_\theta^1/\partial r$ near the boundaries and yet this derivative is used in the Taylor series expansion (10) from which the boundary conditions on v_θ^1 were derived. Since the latter series must converge we can estimate that the analysis is only valid if the Reynolds number Re_d, like $d_1 E/a\nu$, is significantly less than unity. That is to say the Reynolds number Re based on the *whirl amplitude* and a typical mean fluid velocity must be small even though the overall Reynolds number based on the dimension a or b can be very large.

5. Pressure, stresses and forces

The oscillatory component p^1 of the pressure corresponding to the general solution (22) and (23) can be obtained from the basic equations of motion as

$$\frac{p^1}{\rho r} = v_\theta^1\left[\omega-\frac{E}{r^2}-F\right]+2iFv_r^1-i\nu\frac{\partial\zeta^1}{\partial r}. \tag{36}$$

Also, if the functions $T_1(r)$ and $T_2(r)$ are such that the oscillatory stresses (radial normal σ_{rr}^1 and tangential $\sigma_{r\theta}^1$) in the liquid are denoted by

$$\frac{\sigma_{rr}^1}{\rho} = \mathrm{Re}\,[T_1(r)\,e^{i(\theta-\omega t)}], \quad \frac{\sigma_{r\theta}^1}{\rho} = \mathrm{Re}\,[T_2(r)\,e^{i(\theta-\omega t)}] \tag{37}$$

then it transpires that

$$T^*(r) = T_i(r)-iT_2(r) = -rv_\theta^1\left[\omega-\frac{E}{r^2}-F\right]-2irFv_r^1+i\nu\left[r\frac{\partial\zeta^1}{\partial r}-\zeta^1\right]. \tag{38}$$

It will be seen that the evaluation of the total forces on the rotor or stator requires the evaluation of $\overline{T^*(a)}$ and $\overline{T^*(b)}$, where the overbar denotes the complex conjugate. It follows by substitution for ζ^1 that

$$T^*(r) = -rv_\theta^1\left[\omega-\frac{E}{r^2}-F\right]-2irFv_r^1$$
$$+i\nu\left(\frac{i(\omega-F)}{\nu}\right)^{\frac{1}{2}}\left[A\left\{\eta\frac{\partial J_\beta(\eta)}{\partial\eta}-J_\beta(\eta)\right\}+B\left\{\eta\frac{\partial Y_\beta(\eta)}{\partial\eta}-Y_\beta(\eta)\right\}\right], \tag{39}$$

where A and B are evaluated from (24) and (25). In order to evaluate the stresses on the rotor and stator surfaces it is necessary to proceed with care along lines similar to those used in developing the boundary conditions at the solid surfaces. First note that to $O(d)$ the stresses σ_{nn} and σ_{ns}, respectively normal to and tangential to the solid surface at point A (figure 2), are given by

$$\sigma_{nn} = \sigma_{rr} + 2d_1 a^{-1}\sigma_{r\theta}\sin(\theta^* - \omega t), \tag{40}$$

$$\sigma_{ns} = \sigma_{r\theta} - d_1 a^{-1}(\sigma_{rr} - \sigma_{\theta\theta})\sin(\theta^* - \omega t). \tag{41}$$

Since $\sigma^0_{rr} - \sigma^0_{\theta\theta} = 0$ the only additional $O(d)$ term is that in the first equation. Furthermore the σ_{rr} and $\sigma_{r\theta}$ terms on the right-hand side must be evaluated at the point A, whose co-ordinates are

$$r = a + d_1\cos(\theta^* - \omega t), \quad \theta = \theta^* - d_1\sin(\theta^* - \omega t)/a.$$

Finally, integrating σ_{nn} and σ_{ns} over the circumference of the rotor and proceeding along similar lines for the stator one obtains the forces on the rotor ($\mathbf{F}_1$) and on the stator ($\mathbf{F}_2$) per unit axial length as

$$\mathbf{F}_1 = a\int_0^{2\pi}(\sigma_{nn} + i\sigma_{ns})\, e^{i\theta^*}\, d\theta^* = \pi a\rho\left[\overline{T^*(a)} - \frac{V_a^2 d_1}{a}\right] e^{i\omega t}, \tag{42}$$

where the overbar denotes the complex conjugate, and

$$\mathbf{F}_2 = -\pi b\rho[\overline{T^*(b)} - V_b^2 d_2/b]\, e^{i(\omega t+\phi)}. \tag{43}$$

Thus (42) and (43) along with (39) represent the principal results of practical importance to the design engineer. It is convenient to define non-dimensional force coefficients f_1 and f_2, which are useful in the presentation and discussion of results:

$$f_1 = \mathbf{F}_1/\pi\rho a^2\omega^2 d_1 e^{i\omega t}, \quad f_2 = \mathbf{F}_2/\pi\rho a^2\omega^2 d_1 e^{i\omega t}. \tag{44}$$

These represent the forces on the inner and outer cylinders per unit deflexion of the *inner* cylinder divided by $\pi\rho a^2\omega^2$. The real parts of f_1 and f_2 represent the forces *in the direction* of the rotor displacement and can be interpreted as added-mass effects. Indeed the added mass M_A of the inner cylinder per unit axial length is simply $M_A = \rho\pi a^2\,\mathrm{Re}\,(f_1)$. Note that since the added mass per unit length of a cylinder starting from rest in an infinite domain of fluid is $\pi\rho a^2$ the factor f_1 represents the added mass relative to this classic result.

The imaginary part of f_1 (or f_2) represents the force perpendicular to the direction of displacement of the rotor *and* in the local instantaneous direction of whirl rotation. Thus a negative imaginary part of f_1 represents a viscous drag or damping of the whirl motions of the rotor. The orientation of this force coefficient f_1 on the rotor is also indicated diagrammatically in figure 3.

6. Whirl without rotation

When there is no mean rotation of either cylinder, so that there is no mean flow, the analysis is considerably simplified since $\beta \equiv 1$. As mentioned and illustrated in the introduction this is termed whirl without rotation. The induced forces, though more readily obtained by setting the mean flow equal to zero prior

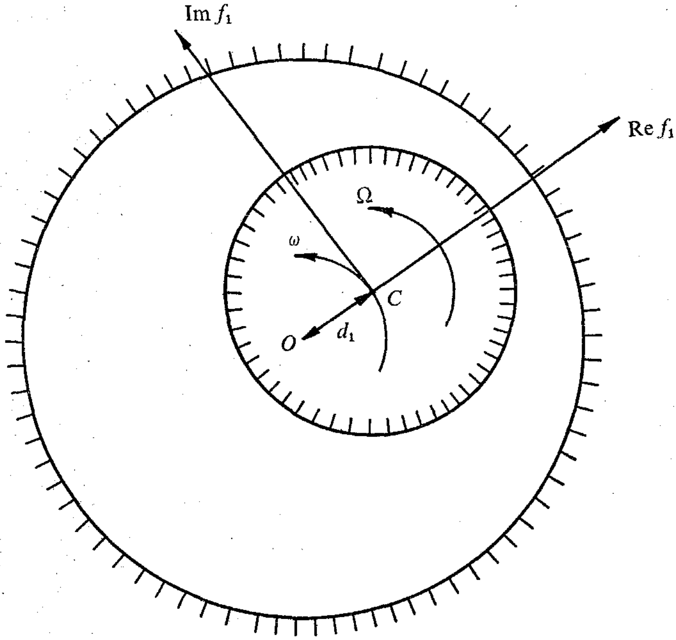

FIGURE 3. Diagram indicating the components of the force coefficient f_1 on the rotor.

to the analysis, can also be deduced from (42) and (43) by setting $E = F = 0$. If γ denotes the ratio of the radii, $\gamma = b/a$, then one obtains

$$\mathbf{F}_1 = \pi\rho a^2\omega^2[-d_1 + \bar{G}(d_1 - d_2 e^{i\phi})]\, e^{i\omega t}, \tag{45}$$

$$\mathbf{F}_2 = -\pi\rho b^2\omega^2[-d_2 e^{i\phi} + \bar{G}\gamma^{-2}(d_1 - d_2 e^{i\phi})]\, e^{i\omega t}, \tag{46}$$

with

$$G = -2W_{22}\bigg/\left[\frac{8}{\pi i R\gamma^2} + W_{02} - \frac{W_{20}}{\gamma^2}\right], \tag{47}$$

where the oscillatory Reynolds number R is defined as $\omega a^2/\nu$ and

$$W_{mn} = J_m\{(iR)^{\frac{1}{2}}\}\, Y_n\{\gamma(iR)^{\frac{1}{2}}\} - J_n\{\gamma(iR)^{\frac{1}{2}}\}\, Y_m\{(iR)^{\frac{1}{2}}\}. \tag{48}$$

Particular calculations were made only for the case $d_2 = 0$, in which the outer cylinder does not deflect (extension of the results to non-zero values of d_2 is straightforward). Then the force coefficients are simply given by

$$f_1 = -1 + \bar{G}, \quad f_2 = -\bar{G}.$$

Before discussing the values of $\bar{G}$, f_1 and f_2 computed from (47) and (48) as functions of γ and R and presented in figures 3 and 4, it is instructive to examine the various asymptotic limits of G.

Consider first the case of large Reynolds number, $R \gg 1$. Hankel's asymptotic expansions for Bessel functions with large arguments leads to

$$G \to -2\gamma^2(iR)^{\frac{1}{2}} \sin\{\delta(iR)^{\frac{1}{2}}\}/[4\gamma^{\frac{1}{2}} - (\gamma+1)\,\delta(iR)^{\frac{1}{2}} \sin\{\delta(iR)^{\frac{1}{2}}\}] \tag{49}$$

provided that $\delta^2 R \gg 1$, δ being the non-dimensional gap width, $\delta = \gamma - 1$. Thus when $R \to \infty$, the limiting value is simply

$$G \to 2\gamma^2/(\gamma^2-1), \quad R \gg 1, \quad \delta^2 R \gg 1. \tag{50}$$

It can be readily confirmed that an inviscid analysis of whirl without rotation yields precisely this result. To determine the manner in which the drag force (or imaginary part of G) tends to zero for large R it is necessary to evaluate the second-order term in the expansion of (49). When the gap width is small, so that $\delta \ll 1$ (but still $\delta^2 R \gg 1$), we may return to the original general form for G and expand to obtain

$$\bar{G} \to 1 + \frac{1}{\delta}\left[1 + \frac{\sqrt{2}}{\delta R^{\frac{1}{2}}}\right] - i\left[\frac{\sqrt{2}}{\delta R^{\frac{1}{2}}}\right], \quad R \gg 1, \quad \delta \ll 1, \quad \delta^2 R \gg 1. \tag{51}$$

Note especially that the drag force may become comparable with the added-mass term if δ and R are such that $R \gg 1$, $\delta^2 R \gg 1$ but $\delta^4 R \ll 1$.

A different limiting value is however obtained if $\delta^2 R \ll 1$ $(R \gg 1)$. Then the leading term, which contrasts with (50) and (51), is

$$\bar{G} \to i[-12/\delta^3 R], \quad R \gg 1, \quad \delta \ll 1, \quad \delta^2 R \ll 1 \tag{52}$$

and hence a large drag force dominates since $\delta^3 R$ is always large compared with unity under these conditions.

The asymptotic limit at low Reynolds numbers $(R \ll 1)$ is more difficult to obtain and requires expansion of the Bessel functions up to five terms. The leading two terms, of order R^{-1} and R^0, are

$$\bar{G} = -\frac{4i(\gamma^2+1)}{R\{(\gamma^2+1)\ln\gamma - (\gamma^2-1)\}} + \frac{\{(\gamma^2-1)(\gamma^4+10\gamma^2+1) - 12\gamma^2(\gamma^2+1)\ln\gamma\}}{6\{(\gamma^2+1)\ln\gamma - (\gamma^2-1)\}^2} + O(R), \quad R \ll 1. \tag{53}$$

Hence the dominant terms are a drag force which behaves like R^{-1} and an added-mass term which is independent of R. Furthermore, when the gap width is small $(\delta \ll 1, R \ll 1)$ the limit of (49) becomes

$$G \to -12i/\delta^3 R + 6/5\delta, \quad R \ll 1, \quad \delta \ll 1. \tag{54}$$

Remarkably the drag force limit of magnitude $12/\delta^3 R$ is identical to that of (52) for $\delta \ll 1$, $R \gg 1$ but $\delta^2 R \ll 1$; hence it appears that this limiting value is valid for all R provided that $\delta^2 R \ll 1$. Of course, when $\delta^2 R \gg 1$ equation (51) becomes the proper limit. In addition the added-mass term only differs from that for $R \gg 1$ by the factor 1·2.

The general function G, from which f_1 and f_2 may be easily obtained, was computed numerically from the definition (47) for various values of the ratio of the radii γ and oscillatory Reynolds number R. Results are presented in figures 4 and 5 for $\mathrm{Re}(G-1) = \mathrm{Re} f_1 = \mathrm{Re}(-1-f_2)$ and for $\mathrm{Im}\, G = -\mathrm{Im} f_1 = \mathrm{Im} f_2$ for $\gamma = 1{\cdot}01$, $1{\cdot}1$ and $2{\cdot}0$. The values are plotted against a modified Reynolds number $R^* = (\gamma-1)R = \omega a(b-a)/\nu$ for convenience. Also shown in these figures are the low Reynolds number asymptotic limits given by (53) for both the real and imaginary parts and the high Reynolds number limit for the real part given

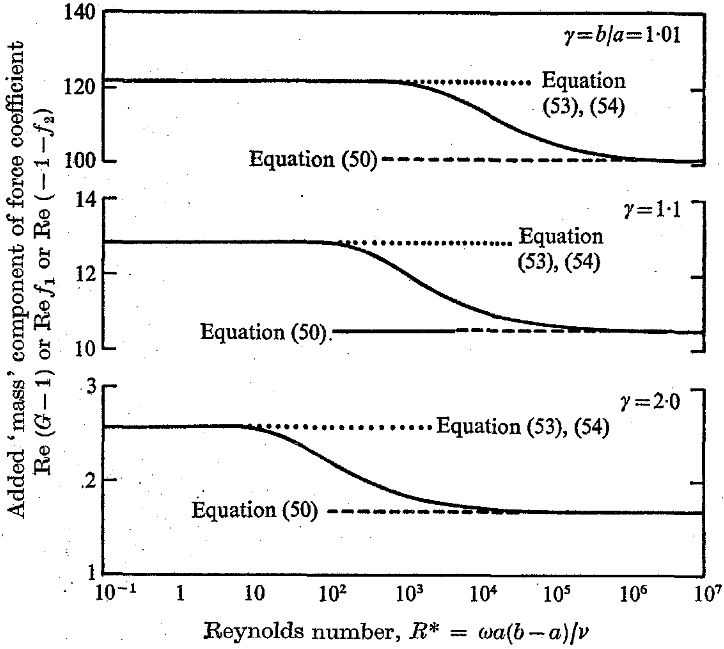

FIGURE 4. Whirl without rotation (with $d_2 = 0$). The 'added mass' component of the force coefficients $\mathrm{Re}(G-1) = \mathrm{Re}\, f_1 = \mathrm{Re}\,(-1-f_2)$ as a function of Reynolds number R^* and non-dimensional radius ratio $\gamma = b/a$, calculated from (47) (solid curves). Also shown are the low and high Reynolds number asymptotes from (53) [or (54)] (dotted curves) and (50) (dashed curves).

by (50). The high Reynolds number limit for the imaginary part for $\delta \ll 1$ given by (52) is also shown in figure 5; the actual asymptote for the cases in which δ is not so small was obtained from the numerical results.

In viewing these results it is clear that virtually all the necessary information from a practical point of view is contained within the asymptotic formulae. It is only necessary to establish the critical value of R or R^* below which the $R \ll 1$ formulae and above which the $R \gg 1$ formulae should be used. It is clear from the figures that this corresponds closely to the point at which the asymptotes cross in figure 5; this point, at least for small δ, is given by $R^* = 72/\delta$. It is noteworthy in this regard that the asymptotic formula (53), which was derived for $R \ll 1$, is actually valid up to values of R much greater than one.

7. Whirl with rotation: laminar mean flow

Let us now turn to the case in which there is a mean flow. For ease of presentation, we shall choose to examine the particular case in which this mean flow is created by rotation of the inner cylinder with angular frequency Ω, while the outer cylinder is at rest in the mean. In synchronous whirl, of course, $\omega = \Omega$; in analyses of journal-bearing lubrication $\omega = 0$. Since there also exist other intermediate types such as half-frequency whirl and since little added difficulty arises

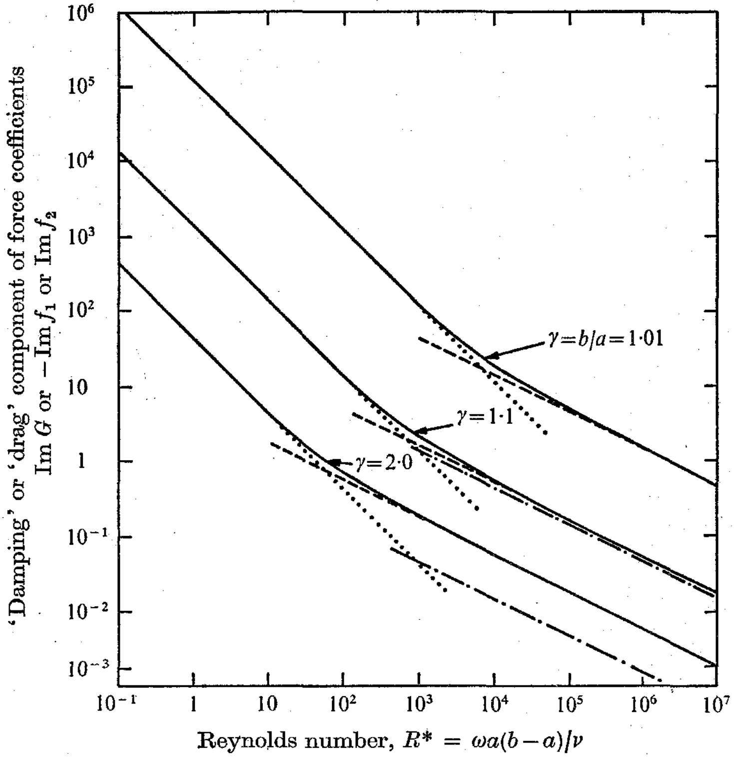

FIGURE 5. Whirl without rotation (with $d_2 = 0$). The 'drag' or 'damping' component of the force coefficients Im $(G) = -\text{Im} f_1 = \text{Im} f_2$ as a function of R^* and γ calculated from (47) (solid curves). Also shown are low and high Reynolds number asymptotes from (53) [or (54)] (dotted curves) and (51) (—·—·—, for small $\gamma - 1$). Actual high R^* asymptotes shown by (– – –).

we shall proceed with the general case $\omega \neq \Omega$. Furthermore we shall concentrate attention on the force coefficient f_1 since this is the primary requirement for analysis of the rotor dynamics and whirl.

Examine first the situation when the mean Couette flow in the annulus is both stable and laminar. In principle, all that is necessary is to set $V_a = a\Omega$ and $V_b = 0$ in (18), so that $E = b^2\Omega/(\gamma^2 - 1)$ and $F = -\Omega(\gamma^2-1)$; then values for the force coefficients f_1 and f_2 could be obtained as functions of γ, the Reynolds number $R = -a\omega^2/\nu$ and Ω/ω once $T^*(a)$ and $T^*(b)$ had been computed from the relation (39). Unfortunately a major difficulty arises in computing the relation (39) in this general case. The calculated answers are extremely sensitive to the accuracy with which one can numerically evaluate the integrals and even with quadruple precision on an IBM 360/75 this could not be done satisfactorily. (N.B. The corresponding calculation of the results for whirl without rotation from (47) can also be quite sensitive to numerical error.) However, on the basis of the experience of the last section, in which all the important trends are contained within the asymptotic formulae, we shall proceed to examine the limiting behaviour in the present case with the objective of delineating the important trends. Furthermore, to reduce the complexity we shall make the simplification that the outer cylinder remains rigid with $d_2 = 0$; the structure of the relations for the forces is such that the complicated terms $T^*(a)$ and $T^*(b)$ always contain the linear coefficient $d_1 - d_2 e^{i\phi}$. Thus extension of the results to cases in which $d_2 \neq 0$ is readily made by observation.

We shall first investigate the asymptotic behaviour of f_1 for lower Reynolds numbers R, so that β is close to unity and the argument η of the Bessel functions is small. Using Hankel's expansions of these Bessel functions and evaluating terms of order R^{-1} and R^0 one finds that

$$f_1 \to -1 + \frac{[4iR^{-1}(\gamma^2-1)\{\Omega/\omega-(1+\gamma^2)\}+\{x_1+x_2(\Omega/\omega)+x_3(\Omega/\omega)^2\}+O(R)]}{[(\gamma^2-1)\{(\gamma^2+1)\ln\gamma-(\gamma^2-1)\}-\frac{1}{24}iR\{x_4+x_5(\Omega/\omega)\}+O(R^2)]}, \quad (55)$$

where $x_1, \ldots, x_5$ are functions of γ only and given by

$$x_1 = \tfrac{1}{3}(\gamma^2-1)^3,$$
$$x_2 = \tfrac{1}{3}(5\gamma^4-\gamma^2+2)+2\gamma^2(\gamma^4+1)(\ln\gamma)/(\gamma^2-1),$$
$$x_3 = -(8\gamma^4-\gamma^2-1)/3(\gamma^2-1)+2\gamma^2(\gamma^4+2\gamma^2-1)(\ln\gamma)/(\gamma^2-1)^2,$$
$$x_4 = 2(1-\gamma^2)(1+4\gamma^2+\gamma^4)(\ln\gamma)+3(\gamma^4-1)(\gamma^2-1),$$
$$x_5 = 3(\gamma^2-1)(5\gamma^2+1)-(17\gamma^4+23\gamma^2+2)(\ln\gamma)+12\gamma^6(\ln\gamma)^2/(\gamma^2-1).$$

Notice that this agrees with the result (53) for whirl without rotation when Ω is set equal to zero. As in that previous case the result (55) is dependent upon $R \ll 1$.

In order to provide some further insight into the results which follow let us digress momentarily and consider the particular case of journal-bearing lubrication, where ω is usually zero. Substituting (55) into the definition (44) of the force coefficient and then letting $\omega \to 0$ one obtains

$$\mathbf{F}_1 = \frac{4\pi\rho a^2\Omega^2 i d_1}{R_m\{(\gamma^2+1)\ln\gamma-(\gamma^2-1)\}} \times \left[1+\frac{R_m x_3}{4i(\gamma^2-1)}+\frac{iR_m x_5}{24(\gamma^2-1)\{(\gamma^2+1)\ln\gamma-(\gamma^2-1)\}}\right],$$

where the Reynolds number $R_m = \Omega a^2/\nu$ is assumed small as in most lubrication analyses. Further, when $\delta = \gamma - 1$ is small this becomes

$$\mathbf{F}_1 \to \frac{6\pi\mu\Omega d_1 i}{\delta^3} - \frac{9\pi\rho a^2\Omega^2 d_1}{4\delta^5}. \quad (56)$$

The first, or dominant, term (provided $R_m \ll \delta^2$) is precisely the force obtained from analyses of journal-bearing lubrication based on the Reynolds equation (e.g. Tipei 1962) provided that the bearing displacement d_1 is small compared with the gap width δ (the most commonly quoted result is actually one half of this owing to the assumption of the Gümbel conditions). Note especially that this represents a force normal to the line of centres and *in the direction of rotation.* As we shall see in the context of whirl analysis, such a force could have a destabilizing effect.

The second term in (56) represents both a force along the line of centres and the first appearance of an inertial term. It is particularly interesting since no analogous term appears to be derived in lubrication analysis from the Reynolds equation. Commonly one finds a viscous force in this direction which is $O(d_1^2)$ and thus not considered in our analysis. Further there has been recent interest in inertial effects in lubrication and Milne (1965) and others have attempted to evaluate these effects by modifying the Reynolds equation; but again the predicted terms are $O(d_1^2)$. The above result based on the Navier–Stokes equations suggests an inertial effect $O(d_1)$, which can be significant especially when R_m approaches δ^2. Note that this force is negative and represents a centring force.

To return to the general case represented by (55) we observe that when the gap width $\delta = \gamma - 1$ is moderately large the leading term in f_1 is clearly

$$f_1 = -\frac{4i\{(1+\gamma^2) - \Omega/\omega\}}{R\{(\gamma^2+1)\ln\gamma - (\gamma^2-1)\}} + O(R^0), \tag{57}$$

and this represents a dominant viscous drag or damping of the whirl motions of the inner cylinder. Assuming $\gamma = O(1)$, it is of interest to note that the typical damping for synchronous whirl ($\Omega \simeq \omega$) is approximately one half of the value of the damping force in the whirl-without-rotation case. Furthermore this 'damping' becomes negative when the whirl frequency ω is less than about one half of the rotation frequency Ω, and of course eventually becomes the principal lubrication force when $\omega \to 0$ (see above). In the context of whirl this suggests the possibilities of whirl instabilities at sub-synchronous frequencies (half-frequency whirl, etc.). The second term $O(R^0)$ represents a force in line with the centres though it may either be positive (like an added-mass effect) or negative (as in the lubrication case) since it is a complicated function of γ and Ω/ω.

As stipulated the above remarks relate to moderately large gap widths δ. When δ becomes small the behaviour of the result (55) is somewhat more complicated. Notice first that for $\delta \ll 1$

$$(\gamma^2-1)\{(\gamma^2+1)\ln\gamma - (\gamma^2-1)\} \to \tfrac{4}{3}\delta^4 + \tfrac{2}{15}\delta^6 + \ldots,$$
$$x_4 \to -\tfrac{32}{15}\delta^6, \quad x_5 \to 12\delta^2 + 24\delta^3,$$
$$x_1 \to \tfrac{8}{3}\delta^3, \quad x_2 \to -\tfrac{8}{3}\delta^3, \quad x_3 \to \tfrac{8}{15}\delta^3.$$

The complications arise in the denominator; if $R = O(\delta^2)$ (assuming $\Omega/\omega = O(1)$) then the first term in the denominator is dominant and the leading real and imaginary parts are

$$f_1 \to -\frac{6i}{\delta^3 R}\left(2 - \frac{\Omega}{\omega}\right) + \frac{9}{4\delta^5}\frac{\Omega}{\omega}\left(2 - \frac{\Omega}{\omega}\right). \tag{58}$$

Thus as in the whirl-without-rotation case [equation (54)] the damping is $O(1/\delta^3 R)$; but the added mass is much larger here, being $O(\delta^{-5})$ instead of $O(\delta^{-1})$.

However there is also another possibility. If $\delta^2 \ll R \ll \delta$ then the x_5 term and not the first term in the denominator predominates and the leading real and imaginary parts are

$$f_1 \to \frac{16}{R^2\delta}\left(\frac{2\omega}{\Omega} - 1\right) - i\frac{128\delta}{3R^3}\left(\frac{2\omega^2}{\Omega^2} - \frac{\omega}{\Omega}\right). \tag{59}$$

Thus, while the result (58) is appropriate for a modified Reynolds number $R^* = \omega a(b-a)/\nu$ which is less than δ^3, (59) would seem a better estimate for $\delta^3 \ll R^* < \delta^2$.

Thus the results for low Reynolds numbers have been obtained in the form of the asymptotic formulae (55), (56), (58) and (59); further comment is postponed until the high Reynolds number or inviscid limit has been considered.

8. Whirl with rotation: high Reynolds numbers

In the last section, the asymptotic behaviour of the general solution at low Reynolds numbers was considered. Useful information can also be obtained in the high Reynolds number or inviscid limit and we turn now to that case. In practice, of course, the mean flow will become unstable to Taylor vortices above a critical Taylor number (Taylor 1935). There is a question as to whether the presence of the additional whirling motions might alter the critical Taylor number; recent stability analyses by Hall (1975) for the case of Couette flow with the inner cylinder rotating at a fluctuating speed and by DiPrima & Stuart (1972*a*, *b*, 1975) for non-coaxial cylinders suggest that the critical Taylor number may only be altered by an amount $O(d^2)$, relatively insignificant in the present context. Furthermore the flow will also become turbulent above some mean flow Reynolds number. The present paper will not attempt to analyse the whirl motions and forces with such complex mean flows. Rather we shall examine the results of our general analysis for a simple mean flow characterized by mean tangential fluid velocities V_a and V_b at the surface of the rotor and stator respectively. Then E and F are given by definition (18) in terms of these velocities and using high Reynolds number expansions similar to those of § 4 one can evaluate $T^*(r)$ from (39) as

$$T^*(r) \to 2rF\left\{\frac{C^*}{r^2}+D^*\right\} - r\left\{\omega - \frac{E}{r^2} - F\right\}\left\{\frac{C^*}{r^2} - D^*\right\}\frac{2(\omega - F)r(d_1 - d_2 e^{-i\phi})}{a^2(\gamma^2-1)}H(r). \tag{60}$$

The first term, written in terms of C^* and D^* [(29) and (30)], is precisely the result one obtains from the inviscid perturbation analysis. The second term contains the function $H(r)$ [equation (32)] and thus $T^*(r)$ has the same non-uniform limiting behaviour as that discussed for v_θ^1 in § 4. The task of finding the limiting behaviour of $T^*(a)$ and $T^*(b)$ for large but finite R is very involved and will not be attempted here. The *inviscid* result does not contain the second term since $H(r) \to 0$ as $R \to \infty$ for $a < r < b$. Hence the *inviscid* perturbation result for the force coefficient f_1 is

$$f_1 = -1 + \frac{2\gamma^2}{(\gamma^2-1)}\left(1 - \frac{V_a}{\omega a}\right)\left(1 - \frac{V_b}{\omega b}\right)\left(1 - \frac{d_2 e^{i\phi}}{d_1}\right). \tag{61}$$

Notice that (61) agrees with the whirl-without-rotation result for higher R when we put $V_a = V_b = 0$. Notice also that for small gap widths and $d_2 = 0$

$$f_1 \to \frac{1}{\delta}\left(1 - \frac{V_a}{\omega a}\right)\left(1 - \frac{V_b}{\omega b}\right). \tag{62}$$

For larger gap widths f_1 from (61) can become negative, especially for frequencies ω which are substantially less than the rotation rate Ω, to which V_a and V_b are related; under these circumstances the fluid actually imposes a force which tends to centre the rotor rather than a decentralizing added-mass force.

The resulting forces in the inviscid limit thus depend, naturally, on the characterization of the mean flow through V_a and V_b. Taylor's (1935) Couette flow data suggest that an approximate characterization would be $V_a \approx \frac{1}{2}\Omega a$ and $V_b \approx \frac{1}{2}\Omega b$. Such values can be used directly in (61) to obtain force coefficients (and added mass). Thus

$$f_1 = -1 + \frac{2\gamma^2}{(\gamma^2-1)}\left(1-\frac{\Omega}{2\omega}\right)^2 \tag{63}$$

and for synchronous whirl ($\Omega = \omega$) the added mass M_A of the rotor per unit axial length is simply

$$M_A = \pi\rho a^2 \,\mathrm{Re} f_1 = \frac{(2-\gamma^2)}{2(\gamma^2-1)}\pi\rho a^2. \tag{64}$$

For small gap widths, $\delta = \gamma - 1 \ll 1$,

$$M_A = \pi\rho a^2/4\delta, \tag{65}$$

which is precisely the added mass for this case estimated by Fritz (1970) on the basis of an approximate lubrication theory analysis.

9. Concluding remarks

In this paper we have studied the fluid motions and forces which occur when annular fluid contained between circular cylinders is set in motion by whirling movements of one or both of the cylinders. When there is no mean Couette flow caused by additional constant rotation of the cylinders (i.e. whirl without rotation) the situation is relatively simple and the forces exerted by the fluid on the cylinders can be computed from (45)–(48). Alternatively, as indicated in figures 4 and 5, the asymptotic formulae yield results which are probably adequate in any practical situation.

When there is a mean Couette flow caused by rotation of one or both of the cylinders a great variety of circumstances can occur. The results for low Reynolds numbers have been obtained in the form of the asymptotic formulae (55), (56), (58) and (59). The result for inviscid perturbations at high Reynolds numbers has also been obtained, in (61), and it was demonstrated that the general solution for perturbations on a steady laminar mean flow does asymptote to this as the Reynolds number tends to infinity. Though these results for whirl with rotation are not as complete as one might wish we shall finally attempt to summarize them in graphs similar to figures 4 and 5.

Confining the illustration to the case of conventional whirl in which $\omega = \Omega$, and considering a rigid stator, $d_2 = 0$, the various results for the 'added mass' component $\mathrm{Re} f_1$ and the 'drag' component $\mathrm{Im} f_1$ are displayed in figure 6, where $\delta\,\mathrm{Re} f_1$ and $-\delta\,\mathrm{Im} f_1$ are plotted against R^* for two small gap widths δ. The lower right corner of figure 6 is repeated on a larger scale in figure 7, where the results are compared with the experimental results of Fritz (1970). Those experimental data are for supercritical Taylor numbers and mostly for turbulent mean flow. They do seem to indicate a trend towards increasing 'added mass' at the lower R^*, a trend which is consistent with the asymptote of (59).

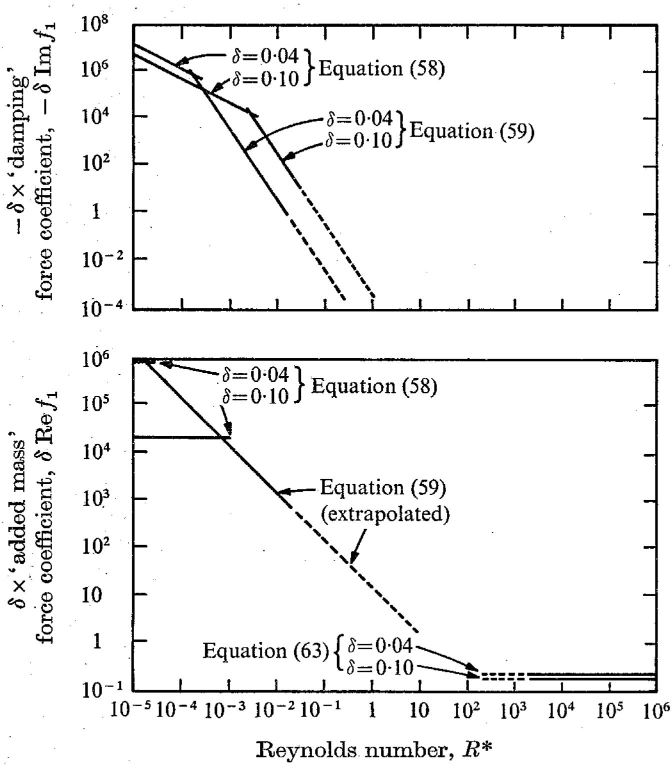

FIGURE 6. Synchronous whirl with rotation (and $d_2 = 0$). The asymptotic behaviour of $\delta\,\mathrm{Re}\,f_1$ and $-\delta\,\mathrm{Im}\,f_1$ as functions of Reynolds number R^* for non-dimensional gap widths δ of 0·04 and 0·1.

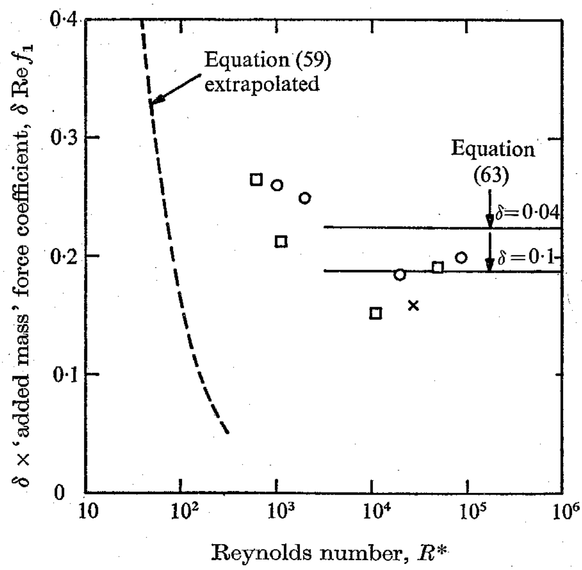

FIGURE 7. The lower right-hand part of figure 6 on a larger scale with experimental results of Fritz (1970): ×, $\delta = 0\cdot04$; □, $\delta = 0\cdot062$; ○, $\delta = 0\cdot1$. The extrapolation of (59) is beyond its estimated region of validity.

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