Solution to Problem 423A:

This problem involves a flow laden with solid particles (for example, sand grains in a river, or dust in the wind). The velocity of the particles is denoted by $v(x,t)$ and is a function of position, x , and time, t. The number density, n (number of particles per unit total volume), may also vary with x and t. Using an elemental volume find the differential equation for n and v which results from application of the principle of conservation of particles, that is to say the statement that particles are neither created nor destroyed.

Consider an elemental Eulerian volume, $dx \times dy \times dz$, and denote the components of the particle velocity, $\underline{v}$, by u, v and w. Then the volume flow into this elemental volume through the side normal to the x direction is $u\ dy\ dz$ and the number of particles flowing in through the same side per unit time is $nu\ dy\ dz$. Moreover the number of particles per unit time flowing out through a parallel side a distance dx away is

$$\left[nu + \frac{\partial (nu)}{\partial x} dx \right]\ dy\ dz \tag{1}$$

and so the net flux of particles through these two sides normal to the x direction is

$$\frac{\partial (nu)}{\partial x} dx\ dy\ dz \tag{2}$$

Therefore considering the other two sets of sides normal to the y and z directions the total net flux of particles out of the elemental control volume is

$$\frac{\partial (nv_i)}{\partial x_i} dx\ dy\ dz \tag{3}$$

and this must be equal to the rate of decrease of particles within the volume if particles are neither created nor destroyed so that

$$\frac{\partial n}{\partial t} + \frac{\partial (nv_i)}{\partial x_i}\ =\ 0 \tag{4}$$

or, in vector terms,

$$\frac{\partial n}{\partial t} + \nabla . (n\underline{v})\ =\ 0 \tag{5}$$