Solution to Problem 422A:

[A] The mixture density, ρ, is given by

$$\rho \;=\; \frac{\text{Total Mass}}{\text{Total Volume}} \;=\; \frac{\rho_A V_A + \rho_L V_L}{V_A + V_L} \;=\; \frac{\rho_A \alpha V + \rho_L (1-\alpha) V}{V} \;=\; \rho_A \alpha + \rho_L (1-\alpha) \tag{1}$$

where ρ_L and ρ_A are respectively the liquid and air densities.

[B] Since $\rho_A \ll \rho_L$, we use $\rho \approx (1-\alpha)\rho_L$. Then neglecting surface tension so that the pressure, p, is the same in both the air and the liquid, that the mixture responds isothermally at the temperature, T, and that the air behaves as a perfect gas so that $p = \rho_A \mathcal{R} T$ it follows that

$$\rho \;=\; \rho_L \left[1 - \frac{\mathcal{R}T/p}{V_L + \mathcal{R}T/p} \right] \tag{2}$$

and therefore

$$p \;=\; \frac{\mathcal{R}T}{V_L} \left[\frac{\rho_L}{\rho_L - \rho} - 1 \right] \tag{3}$$

But by definition, the speed of sound, c, is given by

$$c \;=\; \left[\frac{dp}{d\rho} \right]_T^{1/2} \tag{4}$$

and therefore

$$c \;=\; \left[\frac{p}{\rho_L \alpha (1-\alpha)} \right]^{1/2} \tag{5}$$

Now consider a large reservoir containing a bubbly mixture of void fraction, α_0, at an absolute pressure, p_0. The mixture flows out of the reservoir through a nozzle of throat area, A^*.

[C] We now seek an expression relating the pressure, p, at any point in the nozzle to the void fraction, α, at that point. The expression will include p_0, α_0 and the constant ρ_L. Since the flow is isothermal $p_{A^*} V_{A^*} = p_A V_A = p V_A = \text{constant}$ and therefore

$$\frac{p}{p_0} \;=\; \frac{V_{A_0}}{V_A} \;=\; \frac{V_{A_0}/VT_0}{V_A/VT_0} \;=\; \frac{\alpha_0}{V_A/(V_{A_0} + V_L)} \;=\; \frac{\alpha_0}{V_A/(V_L + pV_A/p_0)} \;=\; \alpha_0 \left[\frac{p}{p_0} + \frac{V_L}{V_A} \right] \tag{6}$$

and therefore

$$\frac{p}{p_0} \;=\; \alpha_0 \left[\frac{p}{p_0} + \frac{(1-\alpha)}{\alpha} \right] \;=\; \frac{\alpha_0}{\alpha} \frac{(1-\alpha)}{(1-\alpha_0)} \tag{7}$$

[D] Integrating the momentum equation for a steady, one-dimensional, frictionless flow $u\, du = -dp/\rho$ using

$$p \;=\; \frac{\alpha_0 p_0}{\alpha} \frac{(1-\alpha)}{(1-\alpha_0)} \tag{8}$$

we find

$$\frac{u^2}{2} = \frac{\alpha_0 p_0}{\rho_L(1-\alpha_0)}\left[\ln\left(\frac{\alpha}{1-\alpha}\right) - \frac{1}{\alpha}\right] + C \tag{9}$$

where C is the integration constant determined by the conditions in the reservoir so that

$$C = \frac{\alpha_0 p_0}{\rho_L(1-\alpha_0)}\left[\frac{1}{\alpha_0} - \ln\left(\frac{\alpha_0}{1-\alpha_0}\right)\right] \tag{10}$$

and

$$\frac{u^2}{2} = \frac{\alpha_0 p_0}{\rho_L(1-\alpha_0)}\left[\ln\left(\frac{\alpha(1-\alpha_0)}{\alpha_0(1-\alpha)}\right) - \frac{1}{\alpha} + \frac{1}{\alpha_0}\right] \tag{11}$$

[E] Assuming the nozzle is choked we seek the relation between α^* and α_0. A choked nozzle implies $u = c$ and therefore

$$\frac{p^*}{\rho_L \alpha^*(1-\alpha^*)} = \frac{\alpha_0 p_0}{\rho_L(1-\alpha_0)}\left[\ln\left(\frac{\alpha^*(1-\alpha_0)}{\alpha_0(1-\alpha^*)}\right) - \frac{1}{\alpha^*} + \frac{1}{\alpha_0}\right] \tag{12}$$

and with

$$\frac{p^*}{p_0} = \frac{\alpha_0}{\alpha^*}\frac{(1-\alpha^*)}{(1-\alpha_0)} \tag{13}$$

it follows that the relation between α^* and α_0 is

$$2(\alpha^*)^2\left[\ln\left(\frac{\alpha^*(1-\alpha_0)}{\alpha_0(1-\alpha^*)}\right) - \frac{1}{\alpha^*} + \frac{1}{\alpha_0}\right] = 1 \tag{14}$$