Solution to Problem 421A:

Liquid enters the boiler tube at $x = 0$ with a velocity U and zero vapor volume fraction. It is also at the boiling point for the prevailing temperature. Heat is supplied to the tube at a steady, uniform rate denoted by Q units of heat per unit time and per unit length of the tube. As the liquid progresses along the tube vapor is formed and the vapor volume fraction, α, increases. We seek expressions for the vapor volume fraction, α, and the mixture velocity, u, as functions of the distance, x, from the inlet.

Consider control volume comprising a small length, dx, of the tube. The heat supplied to this control volume per unit time is $Q\ dx$. Therefore the mass of vapor produced in the control volume per unit time is $Q\ dx/\mathcal{L}$ and therefore the volume of vapor produced per unit time is $Q\ dx/\rho_V\mathcal{L}$. But the volume rate of vapor entering the control volume is αuA and the volume rate of vapor leaving the control volume is

$$\alpha uA \;+\; A\frac{d(\alpha u)}{dx}dx \tag{1}$$

We are given (1) that the temperature and density, ρ_V, of the vapor are constant, (2) that all the liquid and the vapor bubbles travel at the same velocity, u, at each axial location, x, (3) that all the heat supplied goes toward vaporization and (4) that, although the mass of liquid vaporized must be included in determining the increase of the vapor volume fraction along the tube, that same mass is negligible in so far as the conservation of liquid volume flow is concerned. As a consequence, conservation of vapor in the control volume requires that

$$\left(\alpha uA + A\frac{d(\alpha u)}{dx}dx\right) - \alpha uA \;=\; \frac{Q\ dx}{\rho_V\mathcal{L}} \tag{2}$$

and therefore

$$\frac{d(\alpha u)}{dx} \;=\; \frac{Q}{\rho_V A\mathcal{L}} \tag{3}$$

so that by integration

$$\alpha u \;=\; \frac{Q}{\rho_V A\mathcal{L}}\ x + \text{constant} \tag{4}$$

But at $x = 0$, $\alpha = 0$ and therefore

$$\alpha u \;=\; \frac{Q}{\rho_V A\mathcal{L}}\ x \tag{5}$$

But, also conservation of the volume (or mass) of the liquid applied to the control volume utilizes the volume flow rate of liquid entering the control volume:

$$(1-\alpha)uA \tag{6}$$

and the volume flow rate of liquid leaving the control volume:

$$(1-\alpha)uA + \frac{d}{dx}\left((1-\alpha)uA\right)\ dx \tag{7}$$

and therefore, neglecting the mass of liquid vaporized

$$\frac{d}{dx}\left((1-\alpha)uA\right) \;=\; 0 \tag{8}$$

so that by integration and using the entrance boundary condition that at $x = 0$, $\alpha = 0$ and $u = U$:

$$u - \alpha u \;=\; U \tag{9}$$

Solving the equations

$$\alpha u \;=\; \frac{Q}{\rho_V A \mathcal{L}}\, x \quad \text{and} \quad u - \alpha u \;=\; U \tag{10}$$

we get the answers:

$$u \;=\; U + \frac{Q}{\rho_V A \mathcal{L}}\, x \quad \text{and} \quad \alpha \;=\; \frac{Q}{\rho_V A \mathcal{L}\;\; [U + Qx/\rho_V A \mathcal{L}]} \tag{11}$$