Solution to Problem 401A:

[1] Since the mass of insoluble gas in the bubble is m and the volume of a bubble of radius, R, is $4\pi R^3/3$, the density of the insoluble gas is $3m/4\pi R^3$. Therefore if the temperature of the bubble is T and the insoluble gas constant is $\mathcal{R}$ it follows that the partial pressure, p_G, of insoluble gas in the bubble and the total pressure in the bubble, p_B, are

$$p_G = \frac{3m\mathcal{R}T}{4\pi R^3} \quad \text{and} \quad p_B = p_V + \frac{3m\mathcal{R}T}{4\pi R^3} \tag{1}$$

It follows from the Rayleigh-Plesset equation that

$$p_V + \frac{3m\mathcal{R}T}{4\pi R^3} - p_\infty = \rho_L R \frac{d^2R}{dt^2} + \frac{3}{2}\rho_L \left(\frac{dR}{dt}\right)^2 + \frac{2S}{R} \tag{2}$$

The equilibrium radius, R_E, is the solution of this equation when $d^2R/dt^2 = 0$ and $dR/dt = 0$, in other words

$$p_V + \frac{3m\mathcal{R}T}{4\pi R_E^3} - p_\infty = \frac{2S}{R_E} \tag{3}$$

so the cubic equation that must be solved for R_E is

$$R_E^3(p_V - p_\infty) - 2SR_E^2 + \frac{3m\mathcal{R}T}{4\pi} = 0 \tag{4}$$

[2] To consider the stability of this equilibrium, we consider what happens when we force this bubble to a slightly larger radius, $R = R_E + \Delta R$, hold it there and then release it at time $t = 0$ when $dR/dt = 0$. Then according to the Rayleigh-Plesset equation

$$\rho_L(R_E + \Delta R)\left(\frac{d^2R}{dt^2}\right)_{t=0} = p_V + \frac{3m\mathcal{R}T}{4\pi(R_E + \Delta R)^3} - p_\infty - \frac{2S}{R_E} \tag{5}$$

and since we assume $\Delta R \ll R_E$ and neglecting all terms of order $(\Delta R)^2$ or higher

$$\rho_L(R_E + \Delta R)\left(\frac{d^2R}{dt^2}\right)_{t=0} \approx \frac{\Delta R}{R_E}\left[\frac{2S}{R_E} - \frac{9m\mathcal{R}T}{4\pi R_E^3}\right] \tag{6}$$

Therefore if ΔR is positive, then $(d^2R/dt^2)_{t=0}$ will be negative and the bubble will accelerate back toward its equilibrium state if and only if

$$\frac{9m\mathcal{R}T}{4\pi R_E^3} > \frac{2S}{R_E} \tag{7}$$

and hence the equilibrium is stable if and only if

$$R_E < \left(\frac{9m\mathcal{R}T}{8\pi S}\right)^{1/2} \tag{8}$$

Alternatively using the equilibrium equation this can be written as

$$R_E < \frac{4S}{3(p_V - p_\infty)} \tag{9}$$