Solution to Problem 340B:

Consider a small section of the foil of length, dx_1. The supersonic flow on the top or suction surface of the foil has been expanded by the angular turn $d\theta = -dh/dx_1$ and hence the pressure, p_1, on that surface is

$$p_1 \;=\; p_\infty - \frac{\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \left(\frac{dh}{dx_1}\right) \tag{1}$$

using the supersonic theory for small angles of turn where M and p_∞ are the Mach number and pressure of the flow upstream of the foil.

On the other hand the flow on the underside or pressure surface has been compressed by the small angle, $d\theta = +dh/dx_1$. Therefore

$$p_2 \;=\; p_\infty + \frac{\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \left(\frac{dh}{dx_1}\right) \tag{2}$$

and hence the pressure difference across this section of the foil is

$$p_2 - p_1 \;=\; \frac{2\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \left(\frac{dh}{dx_1}\right) \tag{3}$$

and the resulting force, dF, on this section of the foil perpendicular to the foil per unit depth normal to the figure is $(p_2 - p_1)dx_1$. The components of this yield the contributions to the lift, $d\mathcal{L}$, and drag, dD, on this section of the foil:

$$d\mathcal{L} \;=\; dF\cos\left(dh/dx_1\right) \;\approx\; \frac{2\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \left(\frac{dh}{dx_1}\right) dx_1 \tag{4}$$

$$dD \;=\; dF\sin\left(dh/dx_1\right) \;\approx\; \frac{2\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \left(\frac{dh}{dx_1}\right)^2 dx_1 \tag{5}$$

and integrating over the length of the foil, the total lift, $\mathcal{L}$, and drag, D, are

$$\mathcal{L} \;=\; \frac{2\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \int_0^L \left(\frac{dh}{dx_1}\right) dx_1 \tag{6}$$

$$D \;=\; \frac{2\gamma M^2 p_\infty}{(M^2-1)^{1/2}} \int_0^L \left(\frac{dh}{dx_1}\right)^2 dx_1 \tag{7}$$

where L is the chord of the foil.

Finally, since $C_L = 2\mathcal{L}/\rho_\infty U_\infty^2 L$ and $C_D = 2D/\rho_\infty U_\infty^2 L$ and $\rho_\infty U_\infty^2 = \gamma m^2 p_\infty$

$$C_L \;=\; \frac{4}{(M^2-1)^{1/2}} \int_0^L \left(\frac{dh}{dx_1}\right) \frac{dx_1}{L} \;=\; \frac{4}{(M^2-1)^{1/2}} \frac{h(L)}{L} \tag{8}$$

$$C_D \;=\; \frac{4}{(M^2-1)^{1/2}} \int_0^L \left(\frac{dh}{dx_1}\right)^2 \frac{dx_1}{L} \tag{9}$$