Solution to Problem 295C:

In a frame of reference fixed in a whale, the motion of the tail (the "fluke"), depicted in Figure 1, can be described as follows:

1. An oncoming uniform stream of velocity, U, representing the forward speed of the whale (in the direction, x).
2. A "heave" motion of the fluke in the direction, y, perpendicular to the x direction. It is assumed that this motion, in combination with the forward motion, U, produces a sinusoidal trajectory of the center of the fluke (as sketched in Figure 1) such that the center of the fluke moves according to

$$y = h \sin \omega t \tag{Dfd1}$$

where ω is the frequency of the oscillatory motion and h is the amplitude. Consequently the wavelength. λ, of the motion is given by $\lambda = 2\pi U/\omega$. It also follows that the inclination of the trajectory with respect to the x axis (as shown in Figure 1) is θ where

$$\tan\theta = \frac{dy}{dx} = \frac{1}{U}\frac{dy}{dt} = \frac{\omega h}{U}\cos\omega t \tag{Dfd2}$$

so that

$$\cos\theta = \left[1+\frac{\omega^2h^2}{U^2}\cos^2\omega t\right]^{-\frac{1}{2}} \quad \text{and} \quad \sin\theta = \frac{\omega h}{U}\cos\omega t\left[1+\frac{\omega^2h^2}{U^2}\cos^2\omega t\right]^{-\frac{1}{2}} \tag{Dfd3}$$

3. A oscillatory rotation of the fluke about its center (or pitching motion) such that the inclination of the fluke relative to the x direction is α where

$$\alpha = \tilde{\alpha}\cos\omega t \tag{Dfd4}$$

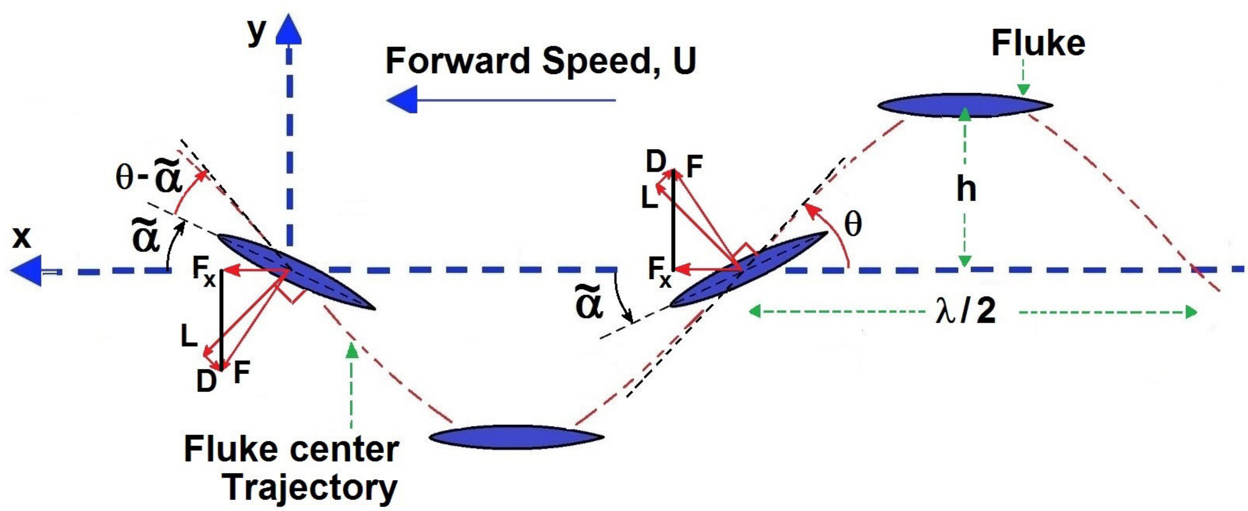

Figure 1: Caudal fin or fluke trajectory and orientation.

We note that the phase relationship of the second and third of these oscillatory motions is such that the relative pitching motion, $\theta - \alpha$, lags the heaving motion by $\pi/2$. This common feature of fish, whale and bird motion is mostly achieved passively by the nature of the structure of the spine, fluke or wing.

Returning to the analysis of fish and cetacea propulsion, it follows that the angle of incidence of the fluke with respect to its *relative* motion through the fluid, α^*, is given by

$$\alpha^* \; = \; \theta - \alpha \; = \arctan\left\{\frac{\omega h}{U}\cos\omega t\right\} - \tilde{\alpha}\cos\omega t \tag{Dfd5}$$

Moreover the magnitude of the velocity of relative motion, V, is given by

$$V^2 \; = \; U^2 + (\frac{dy}{dt})^2 \; = \; U^2 + \omega^2 h^2 \cos^2(\omega t) \tag{Dfd6}$$

and therefore the lift and drag forces acting on the fluke (planform area, A_F), L and D, are given by $\frac{1}{2}\rho V^2 A_F C_L$ and $\frac{1}{2}\rho V^2 A_F C_D$ respectively where C_L and C_D are the lift and drag coefficients for the fluke. It follows that the component of the instantaneous propulsive force in the x direction, F_x, is given by

$$\frac{F_x}{\frac{1}{2}\rho U^2 A_F} \; = \; \left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}\{C_L \sin\theta - C_D\cos\theta\} \; = \; \left\{C_L(\frac{\omega h}{U})\cos\omega t - C_D\right\}\left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}} \tag{Dfd7}$$

The force in the y direction normal to the forward motion, F_y, is given by

$$\frac{F_y}{\frac{1}{2}\rho U^2 A_F} \; = \; \left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}\{C_L \cos\theta + C_D\sin\theta\} \; = \; \left\{C_L + C_D(\frac{\omega h}{U})\cos\omega t\right\}\left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}} \tag{Dfd8}$$

Clearly by symmetry the average of F_y over one cycle is zero and this is appropriate for a neutrally buoyant fish. However, a non-neutrally buoyant fish might require a non-zero lateral force in order to balance its weight or buoyancy. This can be achieved by adding in an additional constant angle of incidence as we will do in the context of bird flight in section (Dfe).

The thrust, T, is given by the mean value of F_x averaged over one cycle of the oscillation of the fluke and therefore

$$T^* \; = \; \frac{T}{\frac{1}{2}\rho U^2 A_F} \; = \; \frac{1}{2\pi}\int_0^{2\pi}\left\{C_L(\frac{\omega h}{U})\cos\omega t - C_D\right\}\left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}} \; d(\omega t) \tag{Dfd9}$$

Before going any further it is evident that the thrust produced by the motion of the fluke is due to the component of the lift force in the x direction which is positive during the entire fluke cycle (except at $y = \pm\pi/2$ where it becomes zero). It is also clear that the drag, D, on the fluke will decrease that propulsive force.

To evaluate the integral in equation (Dfd8) it remains to specify C_L and $C_D$which, if we assume that the oscillatory motions are sufficiently slow that quasisteady coefficients can be used, will be functions primarily of the local angle of incidence, $\alpha^* = \theta - \alpha$. For the purposes of this demonstration we will neglect the drag and assume the lift coefficient is the same as that of a flat plate namely

$$C_L \approx 2\pi\sin\alpha^* \; = \; 2\pi\sin\{\theta - \tilde{\alpha}\cos\omega t\} \tag{Dfd10}$$

which, using the expressions (Dfd3), becomes

$$C_L \; = \; 2\pi\left[\frac{\omega h}{U}\cos\omega t\cos\{\tilde{\alpha}\cos\omega t\} - \sin\{\tilde{\alpha}\cos\omega t\}\right]\left[1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right]^{-\frac{1}{2}} \tag{Dfd11}$$

Substituting this into equation (Dfd8) and setting $C_D = 0$ yields

$$T^* = \frac{T}{\frac{1}{2}\rho U^2 A_F} = \frac{\omega h}{U}\int_0^{2\pi}\left[\frac{\omega h}{U}\cos\omega t\cos\{\tilde{\alpha}\cos\omega t\} - \sin\{\tilde{\alpha}\cos\omega t\}\right]\cos\omega t\; d(\omega t) \tag{Dfd12}$$

Note that the dimensionless thrust, T^*, is a function only of the two dimensionless parameters, $\omega h/U$and $\tilde{\alpha}$. If the angle $\tilde{\alpha}$ is small the the above expression may be approximately evaluated as

$$T^* \;\approx\; \pi\frac{\omega h}{U}\left\{\frac{\omega h}{U} - \tilde{\alpha}\right\} \tag{Dfd13}$$

Note that T is zero when the transient effective angle of incidence, $\theta - \tilde{\alpha}$, is zero and the foil follows the sinusoidal path without any inclination to it. Note also that, perhaps surprisingly, the thrust takes the substantial but characteristic value of $\pi\omega^2 h^2/U^2$ when $\tilde{\alpha} = 0$ and the heave motion occurs with zero inclination of the foil to the direction of motion. Also note that a negative value of $\tilde{\alpha}$ yields an increased thrust though there will be a limit to this when the maximum transient angle of incidence, $\theta - \alpha$, exceeds the value at which the foil stalls.

The thrust, T, will be balanced by the drag on the entire fish (or cetacean), D_B, so that

$$T \;=\; D_B \;=\; \frac{1}{2}\rho A_B U^2 C_{DB} \tag{Dfd14}$$

where C_{DB} is the drag coefficient for the entire fish based on an area, A_B, which might be conveniently chosen as the maximum cross-sectional area of the fish normal to the direction of motion. It follows that the swimming velocity of the animal, U, is given by the solution of $T^* = C_{DB}A_B/A_F$ and this yields

$$U \;=\; \frac{\pi\omega h A_F}{2A_B C_{DB}}\left[\left\{\tilde{\alpha}^2 + \frac{4A_B C_{DB}}{\pi A_f}\right\}^{\frac{1}{2}} - \tilde{\alpha}\right] \tag{Dfd15}$$

and, for the characteristic motion in which $\tilde{\alpha} = 0$ this yields

$$U \;=\; \omega h\left\{\frac{\pi A_F}{A_B C_{DB}}\right\}^{\frac{1}{2}} \tag{Dfd16}$$

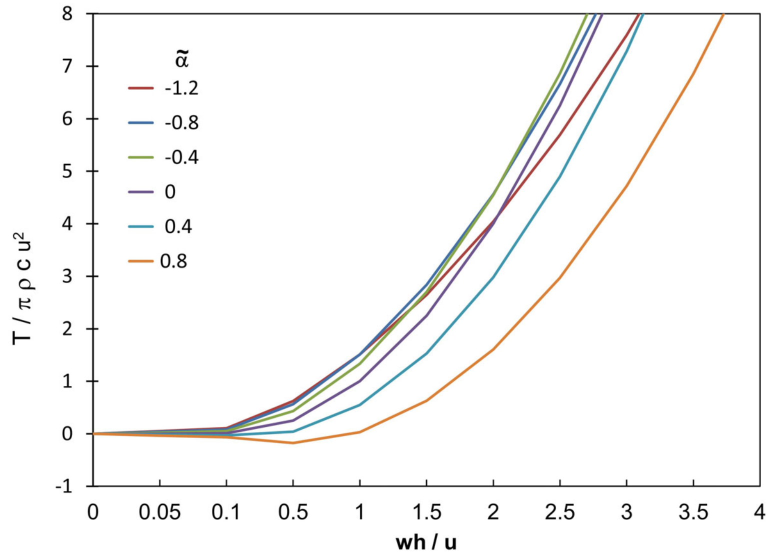

Figure 2: Variation of the dimensionless thrust, $\overline{T}/\pi\rho c U^2$, with dimensionless frequency, $\omega h/U$ for various $\tilde{\alpha}$ (in radians).

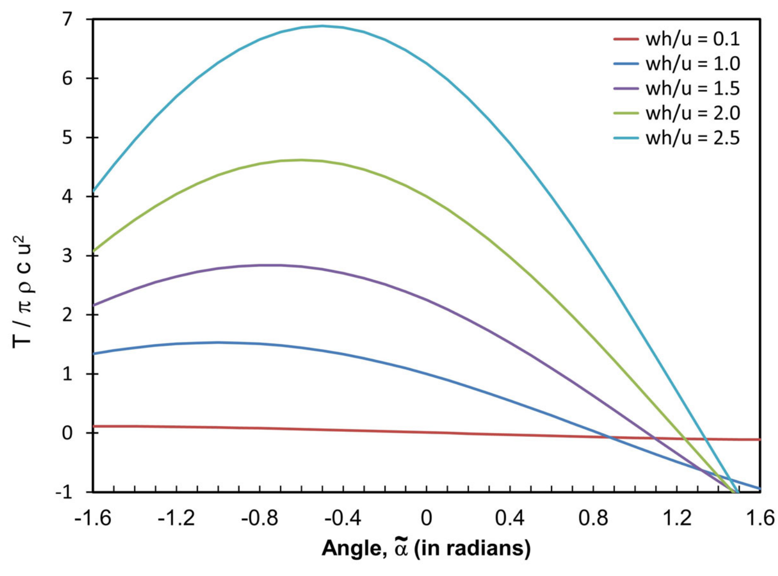

Figure 3: Variation of the dimensionless thrust, $\overline{T}/\pi\rho c U^2$, with the angle, $\tilde{\alpha}$ (in radians).