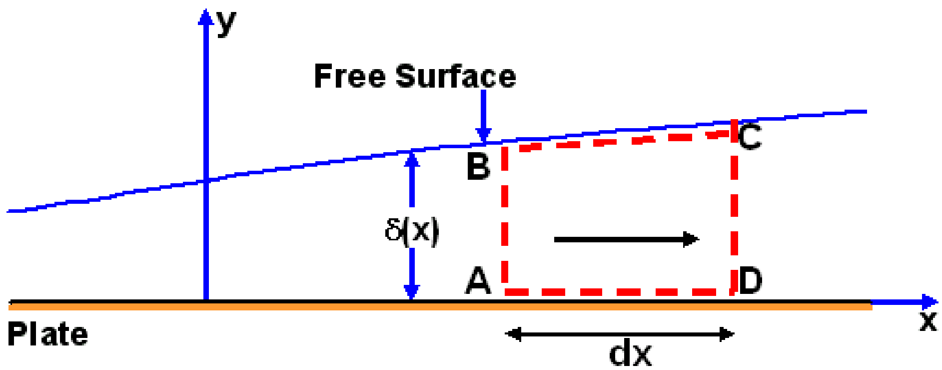
Solution to Problem 250D

A useful control volume for this problem is the volume ABCD of length dx:



The mass flow rate per unit depth perpendicular to the sketch through any plane normal to the plate is

$$m = \int_0^\delta \rho u dy$$

and it follows from continuity that this must be independent of x. It also follows that this can be rewritten in the form

$$m = \rho\delta U \int_0^1 \left(\frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \rho\alpha\delta U$$

where the profile parameter, α, is assumed to be constant. Therefore the only quantities in this last equation which vary with x are δ and U.

Now we prepare to apply the momentum thereom in the x direction. The flux of x-momentum through AB (per unit depth) is given by

$$\int_0^\delta \rho u^2 dy$$

and therefore the flux of x-momentum through CD is

$$\int_0^\delta \rho u^2 dy + dx\frac{d}{dx}\left\{\int_0^\delta \rho u^2 dy\right\}$$

Since there is no mass flow through AD or BC the difference between these two momentum fluxes, the net momentum flux out of the control volume, must be equal to the force on the control volume per unit depth. There is no pressure force since the pressure is uniform everywhere. Therefore the only force is that due to the shear stress acting at the wall and this is equal to

$$\mu dx\left(\frac{du}{dy}\right)_{y=0} = \mu dx\frac{U}{\delta}\left\{\frac{d\left(\frac{u}{U}\right)}{d\left(\frac{y}{\delta}\right)}\right\}_{y=0} = dx\frac{\beta\rho\nu U}{\delta}$$

acting in the negative x direction. Therefore the momentum thereom gives

$$\frac{d}{dx}\left\{\int_0^\delta \rho u^2 dy\right\} = -\frac{\beta\rho\nu U}{\delta}$$

or

$$\rho\gamma\frac{d}{dx}\left\{\delta U^2\right\} = -\frac{\beta\mu U}{\delta}$$

This is a second equation in which the only variables are δ and U. Eliminating U using the equation for the mass flow rate ($U = m/\rho\alpha\delta$) yields

$$\frac{d\delta}{dx} = \frac{\rho\nu\alpha\beta}{m\gamma}$$

and integrating this using the condition that $\delta = \delta_0$ at $x = 0$ yields

$$\delta = \delta_0 + \frac{\rho\nu\alpha\beta}{m\gamma}x$$