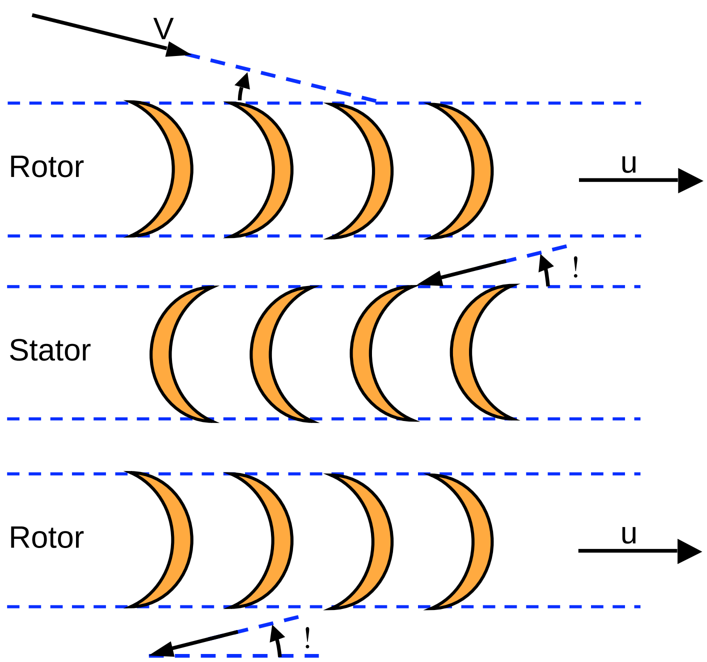
Solution to Problem 225A

This problem requires analysis of the two-stage turbine consisting of a rotor followed by a stator followed by a second rotor: It is to be assumed that all the angles α and β are sufficiently small so that $\cos\alpha$ and



$\cos\beta$ can be approximated by unity. It is also assumed that frictional effects in both the rotors and the stator can be included using the same constant, C, for all three rows of blades where C is defined as follows: Relative velocity leaving blades = $-C\times$ Relative velocity entering blades. We will denote the absolute and relative velocities in the direction of u by v and w respectively and use the subscripts 1 through 6 to denote conditions, respectively, at the inlet to the first rotor, discharge from the first rotor, inlet to the stator, discharge from the stator, inlet to the second rotor and discharge from the second rotor. We will also denote the mass flow rate through all stages in a direction perpendicular to u by $\dot{M}$.

First Rotor: In order to use the steady flow version of the momentum theorem we must utilize a control volume around the first rotor which is moving with the first rotor at the velocity u. Then the velocities relative to that control volume in the u direction are approximately:

$$v_1 = V \qquad \text{and} \qquad w_1 = V - u \qquad \text{and} \qquad w_2 = -Cw_1 = -C(V-u)$$

The net momentum flux in the u direction exiting the first rotor is therefore

$$\dot{M}(w_2 - w_1) = -\dot{M}(V-u)(1+C)$$

and this must be equal to the force on the fluid in the u direction within the first rotor. Consequently the force on the first rotor in the u direction, F_{R1}, is

$$F_{R1} = \dot{M}(V-u)(1+C)$$

[Note that if this was the only stage in a single-rotor turbine, then the "blade efficiency", η_{1rotor}, of that single rotor impulse turbine, defined as the ratio of the power transmitted to the rotor, $F_{R1}u$, to the available energy in the incoming flow, $\dot{M}V^2/2$, would be

$$\eta_{1rotor} = \frac{2F_{R1}u}{\dot{M}V^2} = \frac{2u}{V}(1+C)\left\{1-\frac{u}{V}\right\}$$

For a value of C equal to 0.9 this becomes

$$\eta_{1rotor} = \frac{2u}{V}\left[1.9 - 1.9\frac{u}{V}\right]$$

and we use these results below to compare turbines with various numbers of rotors.]

Stator: It follows that the velocities relative to the stator at inlet to and discharge from that stage in the u direction are approximately:

$$w_3 = v_3 = w_2 + u = u(1+C) - CV \qquad \text{and} \qquad w_4 = v_4 = -Cv_3 = C^2V - uC(1+C)$$

Second Rotor: In order to use the steady flow version of the momentum theorem we must utilize a control volume around the second rotor which is moving with the second rotor at the velocity u. Then the velocities relative to that control volume in the u direction are approximately:

$$w_5 = v_4 - u = -Cv_3 = C^2V - u(1+C+C^2) \qquad \text{and} \qquad w_6 = -Cw_5 = -C^3V + uC(1+C+C^2)$$

The net momentum flux in the u direction exiting the second rotor is therefore

$$\dot{M}(w_6 - w_5) = -\dot{M}[VC^2(1+C) - u(1+2C+2C^2+C^3)]$$

and this must be equal to the force on the fluid in the u direction within the second rotor. Consequently the force on the second rotor in the u direction, F_{R2}, is

$$F_{R2} = \dot{M}[VC^2(1+C) - u(1+2C+2C^2+C^3)]$$

Adding F_{R1} and F_{R2} the total force, F_R, on the rotors in the u direction is therefore

$$F_R = \dot{M}[VC^2(1+C) - u(1+2C+2C^2+C^3) + (V-u)(1+C)]$$

$$F_R = \dot{M}[V(1+C+C^2+C^3) - u(2+3C+2C^2+C^3)]$$

Consequently the power, P, transmitted to the rotor is $P = F_Ru$. The "blade efficiency", η_{2rotor}, is defined as the ratio of this power to the available energy in the incoming flow prior to the first stage namely $\dot{M}V^2/2$ so that the blade efficiency in this case is given by

$$\eta_{2rotor} = \frac{2u}{V}\left[(1+C+C^2+C^3) - (2+3C+2C^2+C^3)\frac{u}{V}\right]$$

For example for a value of C equal to 0.9 this becomes

$$\eta_{2rotor} = \frac{2u}{V}\left[3.239 - 6.849\frac{u}{V}\right]$$

To evaluate a third stage we first need the velocities relative to a second stator which become:

$$w_7 = v_7 = w_6 + u = -C^3V + u[1+C+C^2+C^3] \qquad \text{and} \qquad w_8 = v_8 = -Cv_7 = C^4V - uC[1+C+C^2+C^3]$$

Then the velocities relative to the third rotor become:

$$w_9 = v_8 - u = C^4V - u[1+C+C^2+C^3+C^4] \qquad \text{and} \qquad w_{10} = -Cw_9 = -C^5V + uC[1+C+C^2+C^3+C^4]$$

so that the net momentum flux in the u direction exiting the third rotor is

$$\dot{M}(w_{10} - w_9) = -\dot{M}[VC^2(1+C) - u(1+2C+2C^2+C^3)]$$

and this must be equal to the force on the fluid in the u direction within the second rotor. Consequently the force on the third rotor in the u direction, F_{R3}, is

$$F_{R3} = \dot{M}[VC^4(1+C) - u(1+2C+2C^2+2C^3+2C^4+C^5)]$$

and the total force on a three stage rotor becomes

$$F_R = \dot{M}[V(1+C+C^2+C^3+C^4+C^5) - u(3+5C+4C^2+3C^3+2C^4+C^5)]$$

and the blade efficiency for a three stage impulse turbine, η_{3rotor}, becomes

$$\eta_{3rotor} = \frac{2u}{V}\left[(1+C+C^2+C^3+C^4+C^5) - (3+5C+4C^2+3C^3+2C^4+C^5)\frac{u}{V}\right]$$

which, for a value of C equal to 0.9, becomes

$$\eta_{3rotor} = \frac{2u}{V}\left[4.69 - 14.83\frac{u}{V}\right]$$

Comparing the one, two and three rotor turbine blade efficiencies, for example for $u/V = 0.1$ we find $\eta_{1rotor} = 0.342$, $\eta_{2rotor} = 0.511$, and $\eta_{3rotor} = 0.642$ and therefore the blade efficiciences increase as more rotors extract more energy from the flow.

A more appropriate comparison would be to examine the maximum blade efficiencies for each of these turbines. For this purpose we differentiate the expressions for η_{1rotor}, η_{2rotor}, and η_{3rotor} with respect to u/V and then set those expressions to zero to find the values of u/V at which the blade efficiencies are a maxiumum. Then we evaluate the blade efficiencies at those values of u/V. In the case of $C = 0.9$ this leads to the following results:

	$(u/V)_{max}$	η_{max}
One Rotor	0.500	0.95
Two Rotors	0.236	0.766
Three Rotors	0.158	0.742

Consequently the lighter the load on the turbine (the larger the value of u/V) the fewer the number of stages needed and the higher the blade efficiency. On the other hand for larger loads and lower u/V the greater the number of stages needed to extract the energy from the inlet flow.