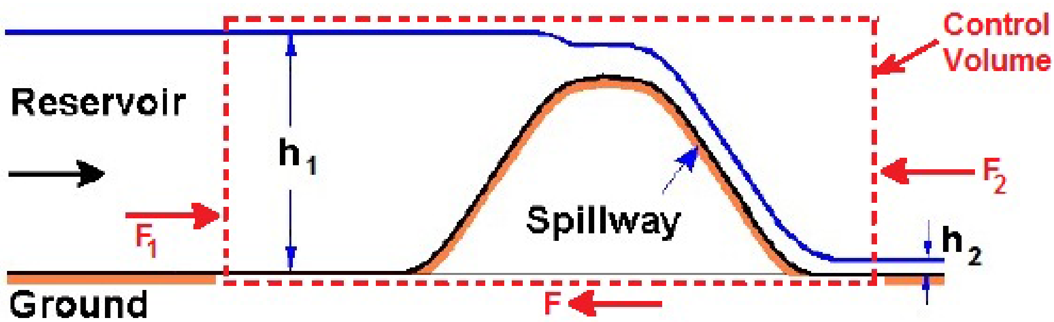
Solution to Problem 220H:

An incompressible, inviscid liquid flow (density, ρ) of depth, h_1, and upstream velocity, V, flows over a spillway:



The depth downstream of the spillway is denoted by h_2. Consider the control volume indicated by the red dashed box in the above figure. Since the fluid is frictionless the forces acting on this control volume are the force F applied to the spillway to hold it in place and the pressure forces (per unit breadth) on the left, F_1, and on the right, F_2, where

$$F_1 \;=\; \int_0^{h_1} p_1(y)\;dy \;=\; p_0 h_1 + \rho g \frac{h_1^2}{2} \quad ; \quad F_2 \;=\; p_0 h_2 + \rho g \frac{h_2^2}{2} \tag{1}$$

where p_0 is the atmospheric pressure above the flow, ρ is the liquid density and g is the acceleration due to gravity. By the momentum theorem the net force (per unit breadth) in the x direction, $F_1 - F_2 - F$, must be equal to the net flux of x momentum out of the control volume or

$$F_1 - F_2 - F \;=\; \frac{1}{2} V_2^2 h_2 \;-\; \frac{1}{2} V_1^2 h_1 \tag{2}$$

In addition the continuity equation requires that

$$\rho V_1 h_1 \;=\; \rho V_2 h_2 \quad \text{so} \quad V_2 = \frac{h_1}{h_2} V_1 \tag{3}$$

and therefore

$$F \;=\; \frac{1}{2}\rho g (h_1 - h_2)\left[h_1 + h_2 - \frac{2h_1}{h_2}\frac{V_1^2}{g}\right] \tag{4}$$

or

$$F \;=\; \frac{1}{2}\rho g h_1 (h_1 - h_2)\left[1 + \frac{h_2}{h_1} - \frac{2h_1}{h_2} Fr\right] \tag{5}$$

where the Fr is the Froude number of the upstream flow, $Fr = V_1^2/gh_1$. Note that the sign of F depends on the Froude number, Fr, and that F is positive for

$$\frac{h_2}{h_1} \;>\; \frac{1}{2}\left[(1 + 8Fr)^{1/2} - 1\right] \tag{6}$$