Solution to Problem 206B:

Consider an increase, x, in the elevation of the fluid inside the tube. Then the decrease in the elevation of the level outside the tube, y, is by continuity:

$$y \quad = \quad A_1 x / A_2 \tag{1}$$

Then by applying the unsteady Bernoulli equation relating the conditions (velocity v and pressure p) at the surface inside the tube (subscript 1) to the conditions at the bottom end of the tube (subscript 2) we obtain

$$\rho(h+x)\frac{\partial v}{\partial t} + (p_2 - p_1) + \rho g(-h-x) = 0 \tag{2}$$

where ρ is the liquid density, g is the acceleration due to gravity. We neglect the terms involving v^2 since they are second order. Furthermore applying the unsteady Bernoulli equation to relate the conditions at the exterior fluid surface (subscript 3) to the conditions at the bottom end of the tube (subscript 2) we obtain

$$(p_3 - p_2) + \rho g(-y+h) = 0 \tag{3}$$

Moreover $p_1 = p_2 = p_a$ where p_a is atmospheric pressure. Combining the above three equations and neglecting the term $x\partial v/\partial t$ since $x << h$:

$$\rho h \frac{dv}{dt} - \rho g(x+y) = 0 \tag{4}$$

since v and x are functions only of t. Furthermore since $v = -dx/dt$ and $y = A_1 x/A_2$:

$$\frac{d^2 x}{dt^2} + \frac{gx(1 + A_1/A_2)}{h} = 0 \tag{5}$$

This equation describes simple harmonic motion and therefore the natural radian frequency of oscillation is

$$\left[\frac{g(1 + A_1/A_2)}{h}\right]^{\frac{1}{2}} \tag{6}$$