Solution to Problem 150B

Since the flow is planar and incompressible the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocity in the vertical direction, v, is zero at the plate and at infinity so it is zero everywhere in the flow. Therefore continuity requires that:

$$\frac{\partial u}{\partial x} = 0$$

so u is only a function of y, $u = u(y)$.

The Navier-Stokes equation in the y-direction reduces to

$$\frac{\partial p}{\partial y} = 0$$

and therefore the pressure can only be a function of x. But the flow must be the same at every x position and consequently the pressure p is uniform everywhere.

The Navier-Stokes equation in the x-direction is:

$$\rho \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Since the flow is planar, $v = 0$, $\partial u/\partial x = 0$, and $\partial p/\partial x = 0$ this becomes:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{d^2 u}{dy^2}$$

We now use separation of variables to solve this PDE by setting:

$$u(y,t) = Y(y)T(t)$$

Substituting this into the PDE leads to an equation with terms that are only functions of y on one side and terms of only t on the other. For this to hold both sides can only be equal to a constant, λ:

$$\frac{1}{T}\frac{dT}{dt} = \frac{\mu}{\rho} Y \frac{d^2 Y}{dy^2} = \lambda$$

The equation for T is then:

$$\frac{dT}{dt} = \lambda T$$

and its solution is:

$$T(t) = c_1 e^{\lambda t}$$

The equation for Y is:

$$\frac{d^2 Y}{dy^2} - \frac{\rho}{\mu} Y = 0$$

and its solution is:

$$Y(y) = c_2 e^{\sqrt{\frac{\rho}{\mu}\lambda} y} + c_3 e^{-\sqrt{\frac{\rho}{\mu}\lambda} y}$$

The boundary condition at the plate is the given relationship for $u(0,t)$:

$$u(0,t) = U(t) = U^* e^{kt}$$

The condition as $y \to \infty$ is that the velocity goes to zero:

$$u(y \to \infty, t) = 0$$

It follows that c_2 must be zero for otherwise the exponential would blow up as $y \to \infty$. Combining the two solutions, $T(t)$ and $Y(y)$, we can write the solution for $u(y,t)$ as

$$u(y,t) = c_4 e^{-\sqrt{\frac{\rho}{\mu}\lambda} y} e^{\lambda t}$$

The given relation for $u(0,t)$ then requires that

$$u(0,t) = c_4 e^{\lambda t} = U^* e^{kt}$$

so that $c_4 = U^*$ and $\lambda = k$. This gives a velocity profile:

$$u(y,t) = U^* e^{kt} e^{-\sqrt{\frac{\rho}{\mu}k} y}$$

We can now calculate the shear stress in the flow:

$$\tau = \mu \frac{\partial u}{\partial y} = -\mu U^* \sqrt{\frac{\rho}{\mu} k} \; e^{kt} e^{-\sqrt{\frac{\rho}{\mu}k} y}$$

and the shear stress at the wall is:

$$\tau\big|_{y=0} = -U^* \sqrt{\rho \mu k} \; e^{kt}$$

This is the shear stress applied to the plate by the fluid. A shear stress of this magnitude in the positive x-direction would be required to give the plate this prescribed motion.