Solution to Problem 138C

First find an expression for $\frac{\partial^2 f}{\partial x^2}$ at the point 0. Find the Taylor series expansion (around 0) for f_1 and f_3:

$$\begin{aligned} f_1 &= f_0 + \frac{h}{1!}\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_0 + O(h^4) \\ f_3 &= f_0 - \frac{h}{1!}\left(\frac{\partial f}{\partial x}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 - \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_0 + O(h^4) \end{aligned}$$

Add the expressions for f_1 and f_3, and solve for $\left(\frac{\partial^2 f}{\partial x^2}\right)_0$:

$$\begin{aligned} f_1 + f_3 &= 2f_0 + h^2\left(\frac{\partial^2 f}{\partial x^2}\right)_0 + O(h^4) \\ \left(\frac{\partial^2 f}{\partial x^2}\right)_0 &= \frac{f_1 + f_3 - 2f_0 + O(h^4)}{h^2} \\ \left(\frac{\partial^2 f}{\partial x^2}\right)_0 &= \frac{f_1 + f_3 - 2f_0}{h^2} + O(h^2) \end{aligned}$$

Now find an expression for $\frac{\partial f}{\partial y}$ at the point 0 using the same method, except this time subtract f_4 from f_2 because we want to keep the $\left(\frac{\partial f}{\partial y}\right)_0$ term.

$$\begin{aligned} f_2 &= f_0 + \frac{h}{1!}\left(\frac{\partial f}{\partial y}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial y^2}\right)_0 + \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial y^3}\right)_0 + O(h^4) \\ f_4 &= f_0 - \frac{h}{1!}\left(\frac{\partial f}{\partial y}\right)_0 + \frac{h^2}{2!}\left(\frac{\partial^2 f}{\partial y^2}\right)_0 - \frac{h^3}{3!}\left(\frac{\partial^3 f}{\partial y^3}\right)_0 + O(h^4) \\ f_2 - f_4 &= 2h\left(\frac{\partial f}{\partial y}\right)_0 + O(h^3) \\ \left(\frac{\partial f}{\partial y}\right)_0 &= \frac{f_2 - f_4 + O(h^3)}{2h} \\ \left(\frac{\partial f}{\partial y}\right)_0 &= \frac{f_2 - f_4}{2h} + O(h^2) \end{aligned}$$

Square the expression for $\frac{\partial f}{\partial y}$ at the point 0, and be sure to keep track of the error terms:

$$\begin{aligned} \left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h} + O(h^2)\right)^2 \\ \left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h}\right)^2 + O(h^4) + \frac{f_2 - f_4}{2h}O(h^2) \\ \left(\frac{\partial f}{\partial y}\right)_0^2 &= \left(\frac{f_2 - f_4}{2h}\right)^2 + O(h) \end{aligned}$$

Putting everything together, and dropping our error terms $O(h)$ and higher (the terms we keep are $O(h^{-2})$):

$$f_0\left(\frac{\partial^2 f}{\partial x^2}\right)_0 = -4\left(\frac{\partial f}{\partial y}\right)_0^2$$

$$f_0\frac{f_1 + f_3 - 2f_0}{h^2} = -4\left(\frac{f_2 - f_4}{2h}\right)^2$$

$$f_0(f_1 + f_3 - 2f_0) = -(f_2 - f_4)^2$$