Solution to Problem 137B:

One of the most powerful tools for the solution of planar potential flows is the method of complex variables. This is based on the so-called Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{1}$$

which have the following mathematical consequence. If we define a complex position vector,

$$z = x + iy = re^{i\theta} \tag{2}$$

and a complex potential $f = \phi + i\psi$ then it follows from the Cauchy-Riemann equations that <u>any</u> function $f(z)$ is necessarily a solution of Laplace's equation

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0 \tag{3}$$

To prove this we replace the independent variables x and y by the variable $z = x + iy$ and its complex conjugate $\overline{z} = x - iy$ so that in general $f(z, \overline{z})$ will be a function of both z and $\overline{z}$. Moreover since

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i} \tag{4}$$

then

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial \overline{z}} = \frac{1}{2} \quad \text{and} \quad \frac{\partial y}{\partial z} = -\frac{\partial y}{\partial \overline{z}} = \frac{1}{2i} \tag{5}$$

If we then examine the derivative:

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \left\{\frac{\partial f}{\partial x}\right\}\left\{\frac{1}{2}\right\} + \left\{\frac{\partial f}{\partial y}\right\}\left\{-\frac{1}{2i}\right\} \tag{6}$$

and therefore

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right\} + \frac{i}{2}\left\{\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right\} \tag{7}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right\} + \frac{i}{2}\left\{\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right\} = 0 \tag{8}$$

because of the Cauchy-Riemann relations. Since $\partial f/\partial \overline{z} = 0$ it follows that f is only a function of z and not of $\overline{z}$. It therefore follows that any function $f(z)$ that satifies the Cauchy-Riemann relations, therefore satisfies $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$ and therefore constitutes the solution to a planar potential flow.