## Solution to Problem 130H

For standing waves confined by vertical walls, the appropriate velocity potential of the fluid motion is of the form

$$\phi = (Ae^{ky} - Be^{-ky})(C\sin kx + D\cos kx)\sin\omega t$$

where $A$, $B$, $C$, $D$, $k$ and $\omega$ are constants to be determined.

The boundary conditions for this specific case are as follows. The conditions in the horizontal direction state that there is no flow through the side walls:

$$BC\#1: \quad u(0,y,t) = 0$$

$$BC\#2: \quad u(L,y,t) = 0$$

In the vertical direction, there will be no flow through the bottom ($y = -H$). The condition at the free surface is more involved. Here we use the kinematic and dynamic (from the unsteady Bernoulli equation) conditions to form the fourth boundary condtion:

$$BC\#3: \quad v(x,-H,t) = 0$$

$$BC\#4a: \quad \frac{\partial h}{\partial t} = v(x,h,t) \approx v(x,0,t)$$

$$Bc\#4b: \quad \left.\frac{\partial \phi}{\partial t}\right|_{y=0} + gh = const$$

To apply the first two boundary conditions, we calculate the velocity in the horizontal direction:

$$u = \frac{\partial \phi}{\partial x} = -k\left(Ae^{ky} + Be^{-ky}\right)(C\cos kx - D\sin kx)\sin\omega t$$

The first boundary condition therefore requires that $C = 0$ and it follows that $D$ can be absorbed into $A$ and $B$ without loss of generality so that

$$\phi = (Ae^{ky} - Be^{-ky})\cos kx \sin\omega t$$

Then

$$BC\#2 \Rightarrow u(L,y,t) = -k\left(Ae^{ky} + Be^{-ky}\right)\sin kL \sin\omega t = 0 \qquad \Rightarrow kL = n\pi, n = 1,2,3,...$$

For the lowest mode, we select n=1, which gives $k = \pi/L$. To apply the second two boundary conditions, we find the velocity in the vertical direction:

$$v = \frac{\partial \phi}{\partial y} = k\left(Ae^{ky} - Be^{-ky}\right)\cos kx \sin\omega t$$

$$BC\#3 \Rightarrow v(x,-H,t) = k\left(Ae^{-kH} - Be^{kH}\right)\cos kx \sin\omega t = 0$$

$$\Rightarrow Ae^{-kH} = Be^{kH} = C$$

$$v = Ck\left[e^{k(y+H)} - e^{-k(y+H)}\right]\cos kx \sin\omega t$$

$$\phi = C\left[e^{k(y+H)} + e^{-k(y+H)}\right]\cos kx \sin\omega t$$

We now apply $BC\#4a$ and integrate to solve for the height of the surface disturbance, $h(x,0,t)$:

$$\frac{\partial h}{\partial t} \approx v(x,0,t) = Ck\left[e^{kH} - e^{-kH}\right]\cos kx \sin\omega t$$

$$h(x,0,t) = -\frac{ck}{\omega}\left[e^{kH} - e^{-kH}\right]\cos kx \cos\omega t$$

To apply $BC\#4b$, we use this expression for h(x,0,t) and calculate $\frac{\partial \phi}{\partial t}$:

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=0} = C\omega\left[e^{kH} + e^{-kH}\right]\cos kx \cos\omega t$$

$$\left.\frac{\partial \phi}{\partial t}\right|_{y=0} + gh = const = C\omega\left[e^{kH} + e^{-kH}\right]\cos kx \cos \omega t - \frac{gCk}{\omega}\left[e^{kH} - e^{-kH}\right]\cos kx \cos \omega t$$

For this relationship to hold for all x,y, and t, the constant must be equal to zero.

$$\Rightarrow \omega^2 = gk\frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}} = gk \tanh kH$$

$$f = \frac{\omega}{2\pi} = \sqrt{\frac{g}{4\pi L}\tanh\frac{\pi H}{L}}$$