Solution to Problem 130B:

As in the case without surface tension we assume a traveling wave solution of the form

$$\phi = Ae^{ky}\sin(kx - \omega t) \tag{1}$$

and the kinematic boundary condition at the surface, $v_{y=0} = \partial h/\partial t$ leads to

$$h \quad = \quad \frac{Ak}{\omega}\cos(kx - \omega t) \tag{2}$$

The effect of surface tension, S, is to modify the dynamic condition on the free surface. The pressure, p, in the water just underneath the water surface is

$$p_{waterundersurface} \quad \approx \quad p_{atmospheric} - S\frac{\partial^2 h}{\partial x^2} \tag{3}$$

since the curvature of the water surface is approximately $\partial^2 h/\partial x^2$ when the amplitude of the waves is small. The sign of this curvature is positive for convex curvature viewed from within the liquid and hence the minus sign on the right hand side of the above equation.

It follows that $p + S\partial^2 h/\partial x^2 = constant$ is the modified dynamic boundary condition and therefore

$$S\left\{\frac{\partial^2 h}{\partial x^2}\right\}_{y=0} - \rho\left\{\frac{\partial \phi}{\partial t}\right\}_{y=0} - \rho g h \quad = \quad 0 \tag{4}$$

is the modified dynamic boundary condition. Substituting for $\left\{\frac{\partial^2 h}{\partial x^2}\right\}$ and $\left\{\frac{\partial \phi}{\partial t}\right\}$ yields

$$\frac{SAk^3}{\rho\omega} + \frac{Akg}{\omega} - \omega A \quad = \quad 0 \tag{5}$$

and therefore

$$c^2 \quad = \quad \frac{\omega^2}{k^2} \quad = \quad \frac{g}{k} + \frac{Sk}{\rho} \tag{6}$$