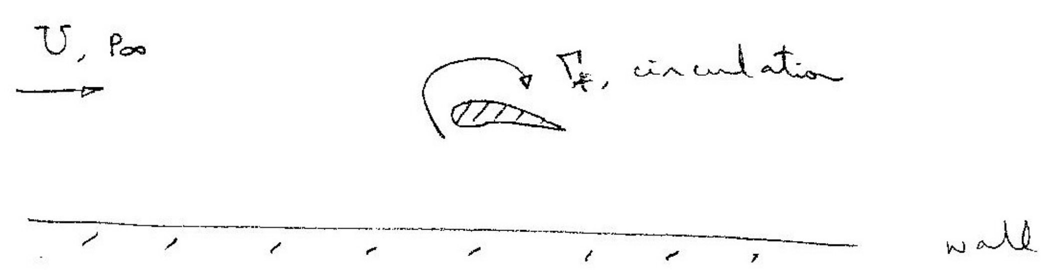
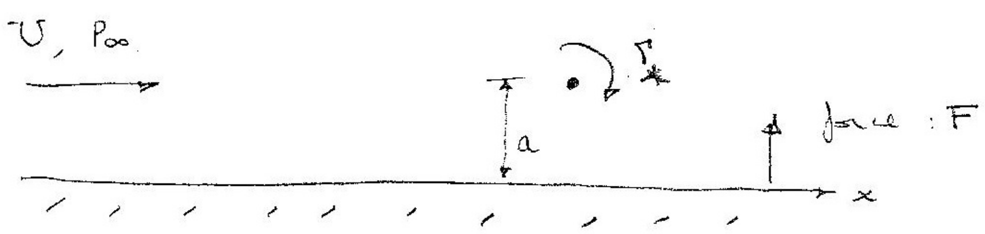
Problem 120H

To a first approximation the potential flow around an airfoil maybe simulated by a simple potential vortex. In this problem we consider some of the consequences of the planar potential flow around an airfoil close to a plane wall by simulating the following flow,



by the following model:



Calculate the force F (per unit depth normal to the sketch), acting on the wall in the upward direction assuming that the pressure underneath the wall is P_∞, the same as in the flow far from the airfoil. Since no net force can act on the body of the fluid, it follows that a downward force of magnitude F acts on the airfoil. Use this result to derive an expression for the lift, L (per unit depth), acting on the foil in terms of ρ the fluid density, Γ the circulation of the vortex, U the free stream velocity, and a the distance from the vortex to the wall. What can you conclude regarding the effect of the neighbouring wall on the lift

We use the method of images to generate a wall, and we find for the stream function

$$\psi = -\frac{\Gamma}{2\pi}\ln\left[x^2+(y-a)^2\right]^{1/2} + Uy + \frac{\Gamma}{2\pi}\ln\left[x^2+(y+a)^2\right]^{1/2}$$

the velocity:

$$u = \frac{\partial\psi}{\partial y} = U - \frac{\Gamma}{2\pi}\left(\frac{y-a}{x^2+(y-a)^2}\right) + \frac{\Gamma}{2\pi}\left(\frac{y+a}{x^2+(y+a)^2}\right)$$

$$u(x, y=0) = U + \frac{\Gamma a}{\pi(x^2+a^2)}$$

at wall $v = -\frac{\partial\psi}{\partial x} = 0$ at $y=0$ (no flow)

The pressure is given from Bernoulli's integral:

$$\frac{2(P_\infty - P)}{\rho} = \left[U + \frac{\Gamma a}{\pi(x^2+a^2)}\right]^2 - U^2$$

$$= \frac{2\Gamma U a}{\pi(x^2+a^2)} + \frac{\Gamma^2 a^2}{\pi^2(x^2+a^2)^2}$$

∴ the force is given by

$$F = \int_{-\infty}^{\infty}(P_\infty - P(x))\,dx = \frac{\rho}{2}\left[\frac{2}{\pi}\Gamma U a\int_{-\infty}^{\infty}\frac{dx}{(a^2+x^2)} + \frac{\Gamma^2 a^2}{\pi^2}\int_{-\infty}^{\infty}\frac{dx}{(a^2+x^2)^2}\right]$$

note $\int_{-\infty}^{\infty}\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_{-\infty}^{\infty} = \frac{1}{a}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{a}$

and note $\frac{\partial}{\partial a}\int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)} = \int_{-\infty}^{\infty}\frac{-2a\,dx}{(x^2+a^2)^2} = -\frac{\pi}{a^2}$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}$$

So we find for F:

$$F = \rho U\Gamma + \frac{\rho}{2}\left(\frac{\Gamma^2 a^2}{\pi^2}\right)\left(\frac{\pi}{2a^3}\right)$$

$$F = \rho U\Gamma\left[1 + \frac{\Gamma}{4\pi a U}\right]$$

wall effect $\to 0$ as $a \to \infty$

normal lift contribution

So the lift must be $L = -F$. The wall increases the lift by amount $\rho\frac{\Gamma^2}{4\pi a}$, which vanishes as the airfoil gets away from the wall.