Solution to Problem 112A

The Lagrangian derivative in three dimensions can be written as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$$

In this problem the flow is steady, so the $\partial/\partial t$ term vanishes. The problem statement also says the velocity in the x-direction is the only velocity we need to consider, so the last two terms on the right hand side of the above equation vanish. As a result, we have

$$\frac{D}{Dt} = u_x \frac{\partial}{\partial x} = u_x \frac{d}{dx}$$

The notation in the problem statement tells you

$$u_x \equiv u = \left(\frac{u_o}{x_o}\right) x$$

We also know that the rate of change of the concentration, c, of chemical constituents *within the fluid* (that is to say travelling with the fluid) is α so that

$$\frac{Dc}{Dt} = \alpha$$

and therefore

$$\frac{Dc}{Dt} = u \frac{dc}{dx} = \alpha$$

so that

$$\alpha = \left(\frac{u_o}{x_o}\right) x \frac{dc}{dx}$$

or

$$\frac{dx}{x} = \frac{u_o}{x_o \alpha} dc$$

$$\int_{x_o}^{x} \frac{dx}{x} = \frac{u_o}{x_o \alpha} \int_{c_o}^{c} dc$$

$$\ln x - \ln x_o = \frac{u_o}{x_o \alpha} (c - c_o)$$

and thus we arrive at

$$c = \frac{\alpha x_o}{u_o} \ln\left(\frac{x}{x_o}\right) + c_o$$