Solution to Problem 109F

Consider a control volume of unit thickness normal to the diagram and bounded by the free surface, two planes perpendicular to the plate a distance ds apart and a surface parallel to the plate that is a distance y below the free surface.

Apply the linear momentum theorem to this control volume in the direction parallel to the plate. Since the velocities and pressures on the ends normal to the plate are identical, they make no net contribution to the momentum flux or the pressure forces parallel to the plate. Therefore the only contributions come from the component of gravity and the shear stress on the boundary at $n = h - y$. Therefore the shear stress on that surface, τ, must be given by

$$\rho g y ds \sin\theta \quad = \quad \rho g (h-n) ds \sin\theta \quad = \quad \tau ds \tag{1}$$

If the flow is laminar $\tau = \mu du_s/dn$ and therefore

$$\rho g (h-n) \sin\theta \quad = \quad \mu \frac{du_s}{dn} \tag{2}$$

Solving for $u_s(n)$

$$u_s(n) \quad = \quad \frac{gn(2h-n)\sin\theta}{2\nu} \tag{3}$$

where we have used the no slip boundary condition that $u_s = 0$ and $n = 0$.