Problem 422A

This problem concerns the flow of a liquid containing air bubbles. The presence of the air bubbles causes the mixture to behave like a compressible fluid. The fraction of the volume of air in a unit volume of mixture is called the ”void fraction” and is commonly denoted by α.

[A] Show that the density of the mixture, ρ, is given by

$$\rho = (1-\alpha)\rho_L + \alpha\rho_A \tag{1}$$

where ρ_L and ρ_A are respectively the liquid and air densities. For the rest of this problem the above will be replaced by

$$\rho = (1-\alpha)\rho_L \tag{2}$$

since $\rho_A \ll \rho_L$ and ρ_L will be regarded as a known constant.

[B] If the air behaves like a perfect gas and if the response of the mixture is isothermal, show that the speed of sound in the mixture, c, is given by

$$c = \left[\frac{p}{\rho_L\alpha(1-\alpha)}\right]^{\frac{1}{2}} \tag{3}$$

where p is the absolute pressure. [Neglect surface tension effects.]

Now consider a large reservoir containing a bubbly mixture of void fraction, α_0, at an absolute pressure, p_0. The mixture flows out of the reservoir through a nozzle of throat area, A^*.

[C] Use the above relations to find an expression relating the pressure, p, at any point in the nozzle to the void fraction, α, at that point. The expression will include p_0, α_0 and the constant ρ_L.

[D] Integrate the momentum equation for a steady, one-dimensional, frictionless flow, namely

$$\rho u\frac{du}{dx} = -\frac{dp}{dx} \tag{4}$$

to find a relation for the velocity, u, of the mixture at any point in the nozzle in terms of the local value of α (as well as p_0, α_0 and ρ_L).

[E] If the nozzle is choked use the results of [C] and [D] to find the relation between the void fraction in the throat, α^*, and that in the reservoir, α_0.

Note:

$$\int \frac{d\alpha}{\alpha(1-\alpha)} = ln\frac{\alpha}{(1-\alpha)} \tag{5}$$

$$\int \frac{d\alpha}{\alpha^2(1-\alpha)} = -\frac{1}{\alpha} + ln\frac{\alpha}{(1-\alpha)} \tag{6}$$