Problem 150F

In cylindrical coordinates, (r, θ, z), the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity, μ, and density, ρ, are

$$\rho \left[\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} \right] = -\frac{\partial p}{\partial r} + f_r + \mu \left[\bigtriangledown^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} \right]$$

$$\rho \left[\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r} \right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + f_\theta + \mu \left[\bigtriangledown^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} \right]$$

$$\rho \frac{Du_z}{Dt} = -\frac{\partial p}{\partial z} + f_z + \mu \bigtriangledown^2 u_z$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinate directions, p is the pressure, f_r, f_θ, f_z are the body force components in the r, θ, z directions and the operators D/Dt and $\bigtriangledown^2$ are

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}$$

$$\bigtriangledown^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

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Find the steady laminar flow in an annulus between an inner cylinder of radius, a, which is at rest and a concentric outer cylinder of radius, b, which is rotating about its axis with angular velocity, Ω. Assume the axial velocity, u_z, is everywhere zero as are all derivatives with respect to z; furthermore assume that $u_r = 0$ and that all derivatives with respect to θ are zero. Find:

[1] the velocity distribution $u_\theta(r)$ within the annulus

[2] the difference in the pressures at the surfaces of the inner and outer cylinders

[3] the shear stresses acting on the surfaces of the inner and outer cylinders

[4] the power required to rotate the outer cylinder

[Note: The general solution of the differential equation:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \frac{X}{r^2} = 0$$

is

$$X = Ar + \frac{B}{r}$$

where A and B are integration constants.]