

## Equations for Conservation of Energy

The third fundamental conservation principle that is utilized in developing the basic equations of fluid mechanics is the principle of conservation of energy. Even in single phase flow the general statement of this principle is complicated when energy transfer processes such as heat conduction and viscous dissipation are included in the analysis. Fortunately it is frequently possible to show that some of these complexities have a negligible effect on the results. For example, one almost always neglects viscous and heat conduction effects in preliminary analyses of gas dynamic flows. In the context of multiphase flows the complexities involved in a general statement of energy conservation are so numerous that it is of little value to attempt such generality. Thus we shall only present a simplified version that neglects, for example, viscous heating and the global conduction of heat (though not the heat transfer from one phase to another).

However these limitations are often minor compared with other difficulties that arise in constructing an energy equation for multiphase flows. In single-phase flows it is usually adequate to assume that the fluid is in an equilibrium thermodynamic state at all points in the flow and that an appropriate thermodynamic constraint (for example, *constant* and *locally uniform* entropy or temperature) may be used to relate the pressure, density, temperature, entropy, etc. In many multiphase flows the different phases and/or components are often *not* in equilibrium and consequently thermodynamic equilibrium arguments that might be appropriate for single phase flows are no longer valid. Under those circumstances it is important to evaluate the heat and mass transfer occurring between the phases and/or components; discussion on this is delayed until the next section (Nbi).

In single phase flow application of the principle of energy conservation to the control volume (CV) uses the following statement of the first law of thermodynamics:

$$\begin{aligned} & \text{Rate of heat addition to the CV, } \mathcal{Q} \\ & + \text{Rate of work done on the CV, } \mathcal{W} \\ & = \\ & \text{Net flux of total internal energy out of CV} \\ & + \text{Rate of increase of total internal energy in CV} \end{aligned}$$

In chemically non-reacting flows the total internal energy per unit mass,  $e^*$ , is the sum of the internal energy,  $e$ , the kinetic energy  $u_i u_i / 2$  ( $u_i$  are the velocity components) and the potential energy  $gz$  (where  $z$  is a coordinate measured in the vertically upward direction):

$$e^* = e + \frac{1}{2} u_i u_i + gz \quad (\text{Nbh1})$$

Consequently the energy equation in single phase flow becomes

$$\frac{\partial}{\partial t} (\rho e^*) + \frac{\partial}{\partial x_i} (\rho e^* u_i) = \mathcal{Q} + \mathcal{W} - \frac{\partial}{\partial x_j} (u_i \sigma_{ij}) \quad (\text{Nbh2})$$

where  $\sigma_{ij}$  is the stress tensor. Then if there is no heat addition to ( $\mathcal{Q} = 0$ ) or external work done on ( $\mathcal{W} = 0$ ) the CV and if the flow is steady with no viscous effects (no deviatoric stresses), the energy equation for single phase flow becomes

$$\frac{\partial}{\partial x_i} \left\{ \rho u_i \left( e^* + \frac{p}{\rho} \right) \right\} = \frac{\partial}{\partial x_i} \{ \rho u_i h^* \} = 0 \quad (\text{Nbh3})$$

where  $h^* = e^* + p/\rho$  is the total enthalpy per unit mass. Thus, when the total enthalpy of the incoming flow is uniform,  $h^*$  is constant everywhere.

Now examine the task of constructing an energy equation for each of the components or phases in a multiphase flow. First, it is necessary to define a total internal energy density,  $e_N^*$ , for each component  $N$  such that

$$e_N^* = e_N + \frac{1}{2}u_{Ni}u_{Ni} + gz \quad (\text{Nbh4})$$

Then an appropriate statement of the first law of thermodynamics for each phase (the individual phase energy equation, IPEE) is as follows:

$$\begin{aligned} & \text{Rate of heat addition to } N \text{ from outside CV, } \mathcal{Q}_N \\ & + \text{Rate of work done to } N \text{ by the exterior surroundings, } \mathcal{W}\mathcal{A}_N \\ & + \text{Rate of heat transfer to } N \text{ within the CV, } \mathcal{Q}\mathcal{I}_N \\ & + \text{Rate of work done to } N \text{ by other components in CV, } \mathcal{W}\mathcal{I}_N \\ & = \\ & \text{Rate of increase of total kinetic energy of } N \text{ in CV} \\ & + \text{Net flux of total internal energy of } N \text{ out of the CV} \end{aligned}$$

where each of the terms is conveniently evaluated for a unit total volume.

First note that the last two terms can be written as

$$\frac{\partial}{\partial t} (\rho_N \alpha_N e_N^*) + \frac{\partial}{\partial x_i} (\rho_N \alpha_N e_N^* u_{Ni}) \quad (\text{Nbh5})$$

Turning then to the upper part of the equation, the first term due to external heating and to conduction of heat from the surroundings into the control volume is left as  $\mathcal{Q}_N$ . The second term contains two contributions: (i) minus the rate of work done by the stresses acting on the component  $N$  on the surface of the control volume and (ii) the rate of external *shaft work*,  $\mathcal{W}_N$ , done on the component  $N$ . In evaluating the first of these, we make the same modification to the control volume as was discussed in the context of the momentum equation; specifically we make small deformations to the control volume so that its boundaries lie wholly within the continuous phase. Then using the continuous phase stress tensor,  $\sigma_{Cij}$ , as defined in equation ?? the expressions for  $\mathcal{W}\mathcal{A}_N$  become:

$$\mathcal{W}\mathcal{A}_C = \mathcal{W}_C + \frac{\partial}{\partial x_j} (u_{Ci} \sigma_{Cij}) \quad \text{and} \quad \mathcal{W}\mathcal{A}_D = \mathcal{W}_D \quad (\text{Nbh6})$$

The individual phase energy equation may then be written as

$$\begin{aligned} & \frac{\partial}{\partial t} (\rho_N \alpha_N e_N^*) + \frac{\partial}{\partial x_i} (\rho_N \alpha_N e_N^* u_{Ni}) = \\ & \mathcal{Q}_N + \mathcal{W}_N + \mathcal{Q}\mathcal{I}_N + \mathcal{W}\mathcal{I}_N + \delta_N \frac{\partial}{\partial x_j} (u_{Ci} \sigma_{Cij}) \end{aligned} \quad (\text{Nbh7})$$

Note that the two terms involving internal exchange of energy between the phases may be combined into an *energy interaction* term given by  $\mathcal{E}_N = \mathcal{Q}\mathcal{I}_N + \mathcal{W}\mathcal{I}_N$ . It follows that

$$\sum_N \mathcal{Q}\mathcal{I}_N = O \quad \text{and} \quad \sum_N \mathcal{W}\mathcal{I}_N = O \quad \text{and} \quad \sum_N \mathcal{E}_N = O \quad (\text{Nbh8})$$

Moreover, the work done terms,  $\mathcal{W}\mathcal{I}_N$ , may clearly be related to the interaction forces,  $\mathcal{F}_{Nk}$ . In a two-phase system with one disperse phase:

$$\mathcal{Q}\mathcal{I}_C = -\mathcal{Q}\mathcal{I}_D \quad \text{and} \quad \mathcal{W}\mathcal{I}_C = -\mathcal{W}\mathcal{I}_D = -u_{Di} \mathcal{F}_{Di} \quad \text{and} \quad \mathcal{E}_C = -\mathcal{E}_D \quad (\text{Nbh9})$$

As with the continuity and momentum equations, the individual phase energy equations can be summed to obtain the combined phase energy equation (CPEE). Then, denoting the total rate of external heat added (per unit total volume) by  $\mathcal{Q}$  and the total rate of external shaft work done (per unit total volume) by  $\mathcal{W}$  where

$$\mathcal{Q} = \sum_N \mathcal{Q}_N \quad \text{and} \quad \mathcal{W} = \sum_N \mathcal{W}_N \quad (\text{Nbh10})$$

the CPEE becomes

$$\frac{\partial}{\partial t} \left( \sum_N \rho_N \alpha_N e_N^* \right) + \frac{\partial}{\partial x_i} \left( -u_{Cj} \sigma_{Cij} + \sum_N \rho_N \alpha_N u_{Ni} e_N^* \right) = \mathcal{Q} + \mathcal{W} \quad (\text{Nbh11})$$

When the left hand sides of the individual or combined phase equations, (Nbh7) and (Nbh11), are expanded and use is made of the continuity equation (Nbb2) and the momentum equation (Nbe5) (in the absence of deviatoric stresses), the results are known as the *thermodynamic* forms of the energy equations. Using the expressions (Nbh9) and the relation

$$e_N = c_{vN} T_N + \text{constant} \quad (\text{Nbh12})$$

between the internal energy,  $e_N$ , the specific heat at constant volume,  $c_{vN}$ , and the temperature,  $T_N$ , of each phase, the thermodynamic form of the IPEE can be written as

$$\begin{aligned} & \rho_N \alpha_N c_{vN} \left\{ \frac{\partial T_N}{\partial t} + u_{Ni} \frac{\partial T_N}{\partial x_i} \right\} = \\ & \delta_N \sigma_{Cij} \frac{\partial u_{Ci}}{\partial x_j} + \mathcal{Q}_N + \mathcal{W}_N + \mathcal{Q} \mathcal{I}_N + \mathcal{F}_{Ni} (u_{Di} - u_{Ni}) - (e_N^* - u_{Ni} u_{Ni}) \mathcal{I}_N \end{aligned} \quad (\text{Nbh13})$$

and, summing these, the thermodynamic form of the CPEE is

$$\begin{aligned} & \sum_N \left\{ \rho_N \alpha_N c_{vN} \left( \frac{\partial T_N}{\partial t} + u_{Ni} \frac{\partial T_N}{\partial x_i} \right) \right\} = \\ & \sigma_{Cij} \frac{\partial u_{Ci}}{\partial x_j} - \mathcal{F}_{Di} (u_{Di} - u_{Ci}) - \mathcal{I}_D (e_D^* - e_C^*) + \sum_N u_{Ni} u_{Ni} \mathcal{I}_N \end{aligned} \quad (\text{Nbh14})$$

In equations (Nbh13) and (Nbh14), it has been assumed that the specific heats,  $c_{vN}$ , can be assumed to be constant and uniform.

Finally we note that the one-dimensional duct flow version of the IPEE, equation (Nbh7), is

$$\frac{\partial}{\partial t} (\rho_N \alpha_N e_N^*) + \frac{1}{A} \frac{\partial}{\partial x} (A \rho_N \alpha_N e_N^* u_N) = \mathcal{Q}_N + \mathcal{W}_N + \mathcal{E}_N - \delta_N \frac{\partial}{\partial x} (p u_C) \quad (\text{Nbh15})$$

where  $A \mathcal{Q}_N$  is the rate of external heat addition to the component  $N$  per unit length of the duct,  $A \mathcal{W}_N$  is the rate of external work done on component  $N$  per unit length of the duct,  $A \mathcal{E}_N$  is the rate of energy transferred to the component  $N$  from the other phases per unit length of the duct and  $p$  is the pressure in the continuous phase neglecting deviatoric stresses. The CPEE, equation (Nbh11), becomes

$$\frac{\partial}{\partial t} \left( \sum_N \rho_N \alpha_N e_N^* \right) + \frac{1}{A} \frac{\partial}{\partial x} \left( \sum_N A \rho_N \alpha_N e_N^* u_N \right) = \mathcal{Q} + \mathcal{W} - \frac{\partial}{\partial x} (p u_C) \quad (\text{Nbh16})$$

where  $A \mathcal{Q}$  is the total rate of external heat addition to the flow per unit length of the duct and  $A \mathcal{W}$  is the total rate of external work done on the flow per unit length of the duct.