Meniscus

Surface energy effects (**surface tension** and **contact angle**) at the junction of a liquid surface and a solid boundary can cause the formation of a meniscus. Consider first the simple case of a single vertical wall bounding a pool of liquid at rest as sketched in Figure 1. The geometry is assumed invariant in the direction normal to the sketch and the liquid surface asymptotes to the horizontal far off to the right.

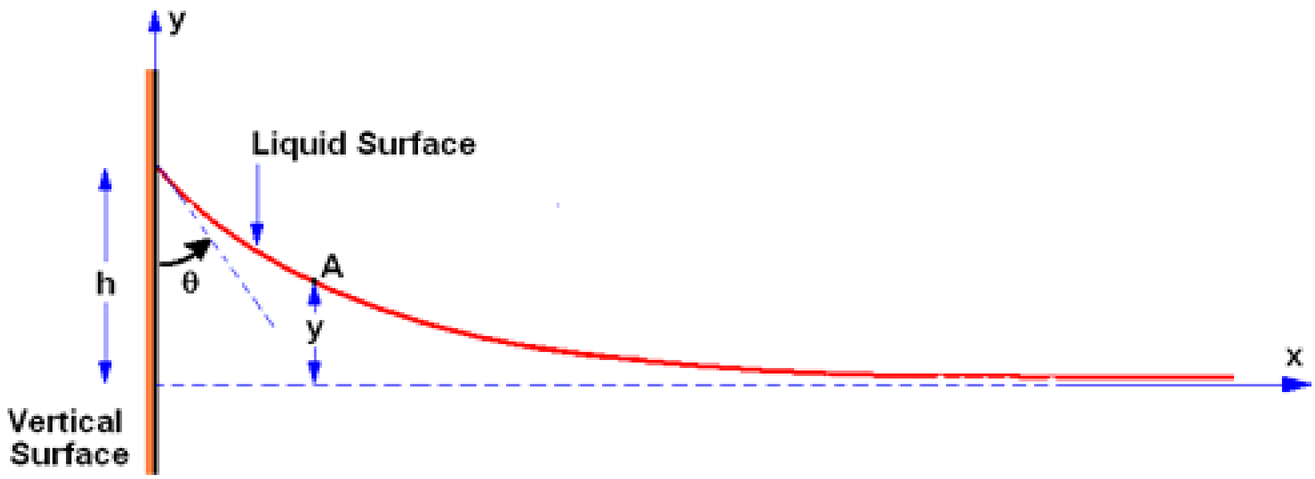

Figure 1: A meniscus at a single vertical wall.

The liquid surface elevation, $y(x)$, is defined so that $y \rightarrow 0$ for large x and the radius of curvature, R, is given by the normal formula:

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{Cp1}$$

so that at a general point such as A the pressure in the liquid at A, p_L, and the pressure in the air at A, p_A, are related by

$$p_A - p_L = \frac{S}{R} \tag{Cp2}$$

But assuming uniform pressure in the air, the pressure in the air at large x must be the same p_A and since the curvature of the surface for large x is zero that must also be the pressure in the liquid just under the surface at large x. Since the pressure in the liquid is the same on a horizontal plane, it follows that the pressure at a point in the liquid at a depth y below the point A must be p_A and therefore the pressure in the liquid at A must also be $p_A - \rho g y$ where ρ is the liquid density and g is the gravitational acceleration. Consequently

$$\frac{S}{R} = \frac{S\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = p_A - (p_A - \rho g y) = \rho g y \tag{Cp3}$$

and therefore the differential equation governing the shape of the liquid surface is

$$\frac{S}{\rho g}\frac{d^2y}{dx^2} = y\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \tag{Cp4}$$

The two boundary conditions which the solution to this differential equation must satisfy are

$$[1] \qquad y \to 0 \quad \text{as} \quad x \to \infty \tag{Cp5}$$

$$[2] \qquad \frac{dy}{dx} = -\cot\theta \quad \text{at} \quad x = 0 \tag{Cp6}$$

where θ is the **contact angle** for that liquid/solid/gas junction.

Clearly the non-linear equation (Cp4) must be solved numerically. However, it is instructive here to detail the approximate solution that results by assuming that the slope of the surface is everywhere small so that the right hand side of equation (Cp4) is approximately y. Then the differential equation becomes

$$\frac{d^2y}{dx^2} = \frac{\rho g y}{S} \tag{Cp7}$$

and the general solution of this equation under the above boundary conditions is

$$y = (S/\rho g)^{\frac{1}{2}} \cot\theta \exp\left(-\{\rho g/S\}^{\frac{1}{2}}\, x\right) \tag{Cp8}$$

Thus the height, h, of the meniscus is given by

$$h = \cot\theta \left[\frac{S}{\rho g}\right]^{\frac{1}{2}} \tag{Cp9}$$

It follows that the height, h, is positive when the contact angle is less than $\pi/2$ (for example for water and glass or metal) and negative when the angle is greater than $\pi/2$ (for example for mercury and glass). It also increases with the surface tension, S.

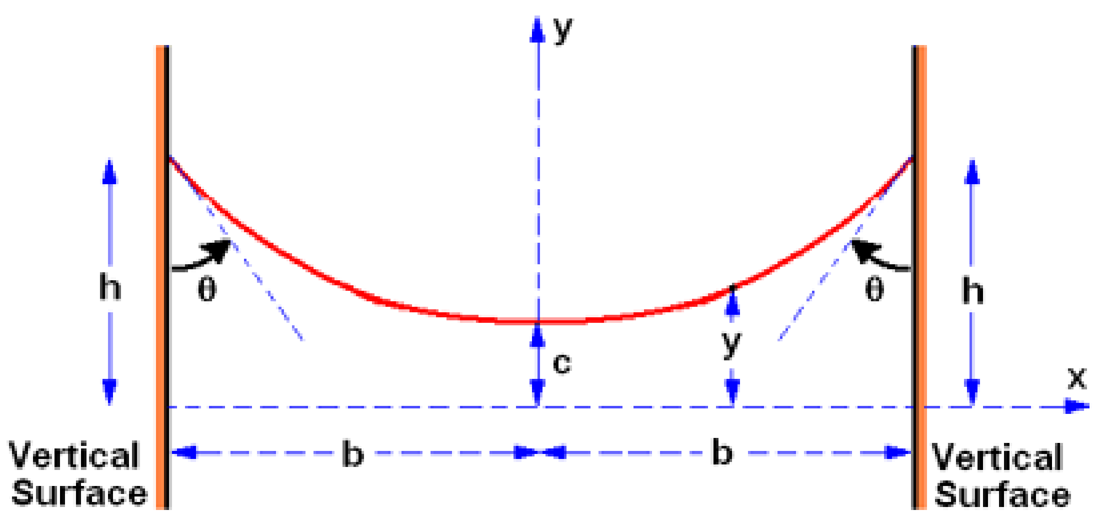

Figure 2: A meniscus at a single vertical wall.

The case of a liquid between two vertical walls (shown in Figure 2) is readily treated by a similar analysis except that the location within the liquid, $y = 0$ at which the liquid pressure is equal to the atmospheric pressure must be determined as part of the solution. The appropriate solution to the approximate differential equation (Cp7) is then

$$y = C_1 \exp\left(-\left\{\frac{\rho g}{S}\right\}^{\frac{1}{2}} x\right) + C_2 \exp\left(+\left\{\frac{\rho g}{S}\right\}^{\frac{1}{2}} x\right) \tag{Cp10}$$

where C_1 and C_2 are constants to be determined by the boundary conditions on the walls, namely

$$[1] \qquad \frac{dy}{dx} = -\cot\theta \quad \text{at} \quad x = -b \tag{Cp11}$$

$$[2] \qquad \frac{dy}{dx} = +\cot\theta \quad \text{at} \quad x = +b \tag{Cp12}$$

From these conditions it follows that

$$C_1 = C_2 = \frac{0.5\,(S/\rho g)^{\frac{1}{2}}\cot\theta}{\sinh\left(\{\rho g/S\}^{\frac{1}{2}}\,b\right)} \tag{Cp13}$$

and the height, h, of the meniscus at the wall above the datum level, $y = 0$, is

$$h = (S/\rho g)^{\frac{1}{2}}\cot\theta\coth\left(\{\rho g/S\}^{\frac{1}{2}}\,b\right) \tag{Cp14}$$

Recalling that $y = 0$ is the level at which the pressure is atmospheric, it follows that the elevation of the center of the meniscus above the atmospheric pressure level, c, is given by

$$c = \frac{(S/\rho g)^{\frac{1}{2}}\cot\theta}{\sinh\left(\{\rho g/S\}^{\frac{1}{2}}\,b\right)} \tag{Cp15}$$

If the two walls were placed in a larger tank of the liquid as shown in Figure 3 the entire meniscus would be elevated above the liquid level far from the walls such that the bottom of the meniscus was a height c above that distant liquid level. This is a simple demonstration of the phenomenon of capillarity in which a liquid confined between walls will rise above the surrounding liquid level provided the contact angle, $\theta < \pi/2$. Conversely it will be depressed if $\theta > \pi/2$.

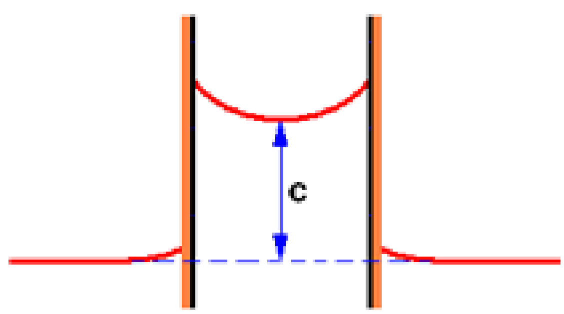

Figure 3: Meniscus between two walls in a larger body of liquid.

We now consider this phenomenon of **capillarity** in more detail and derive results for the elevation or depression of meniscii in narrow tubes which, unlike the above results, are not limited to slopes, dy/dx, which are small.