Atmospheres

In **Fluid Statics**, the vertical gradient of the pressure, p, in a fluid at rest was found to be

$$\frac{dp}{dy} = -\rho g \tag{Ce1}$$

where y is the elevation measured vertically upward, $\rho(y)$ is the fluid density and $g(y)$ is the acceleration due to gravity. Though both $\rho(y)$ and $g(y)$ may be functions of elevation we will consider only atmospheres which are thin compared with the radius of the planet (or other extra-terrestrial object) to which they belong so that g may be considered uniform.

It is convenient to choose sea level (or the fluid surface) as the origin of y and normal atmospheric pressure at sea level (or the surface) as the reference pressure p_0. Then, to proceed with the integration of the above differential equation, we need to know how the density varies within the atmosphere. In the most general circumstances, we recognize that, thermodynamically, ρ is determined knowing the temperature and pressure of the fluid and its **equation of state**. Of these two variables, the temperature and pressure, the temperature variations are the most complex since they depend on many factors including the absorption of solar radiation at different altitudes and the mass and energy exchange processes that occur as a result of mixing between the layers at different altitudes. There are several empirical ways to overcome these complications and we will illustrate two of these methods by reference to two planetary atmospheres, that of the Earth and that of Venus. For reference, we include in Figure 1 data on the U.S. standard atmosphere; this shows an average or standard atmosphere as the variations in temperature and pressure with the altitude. Moreover, the same data is presented in Table 1.

TABLE I. The U.S. Standard Atmosphere.

Altitude (m)	Temperature ($^\circ K$)	Pressure (kPa)	Density (kg/m^3)	Altitude (m)	Temperature ($^\circ K$)	Pressure (kPa)	Density (kg/m^3)
-1000	294.6	111.3	1.135	4500	258.9	57.8	0.777
-500	291.4	107.5	1.285	5000	255.7	54.0	0.736
0	288.2	101.3	1.225	10000	223.3	26.5	0.414
500	284.9	95.5	1.167	11000	216.7	22.7	0.365
1000	281.7	89.9	1.112	11100	216.7	22.4	0.359
1500	278.4	84.5	1.058	15000	216.7	12.1	0.195
2000	275.2	79.5	1.001	20000	216.7	5.3	0.073
2500	271.9	74.7	0.957	25000	221.6	2.6	0.039
3000	268.7	70.1	0.909	30000	226.5	1.2	0.018
3500	265.4	65.8	0.863	35000	236.5	0.6	0.008
4000	262.2	61.7	0.819				

The first way to proceed is to assume that the complications of radiation and mixing result in a given temperature distribution, $T(y)$, specifically the temperature distribution for the Earth's atmosphere in Table 1. Furthermore we know that the gases in the Earth's atmosphere closely follow the **perfect gas law**, $p = \rho \mathcal{R} T$ (where $\mathcal{R}$ is the **gas constant**, so that the density is given by

$$\rho(y) = \frac{p}{\mathcal{R}T(y)} \tag{Ce2}$$

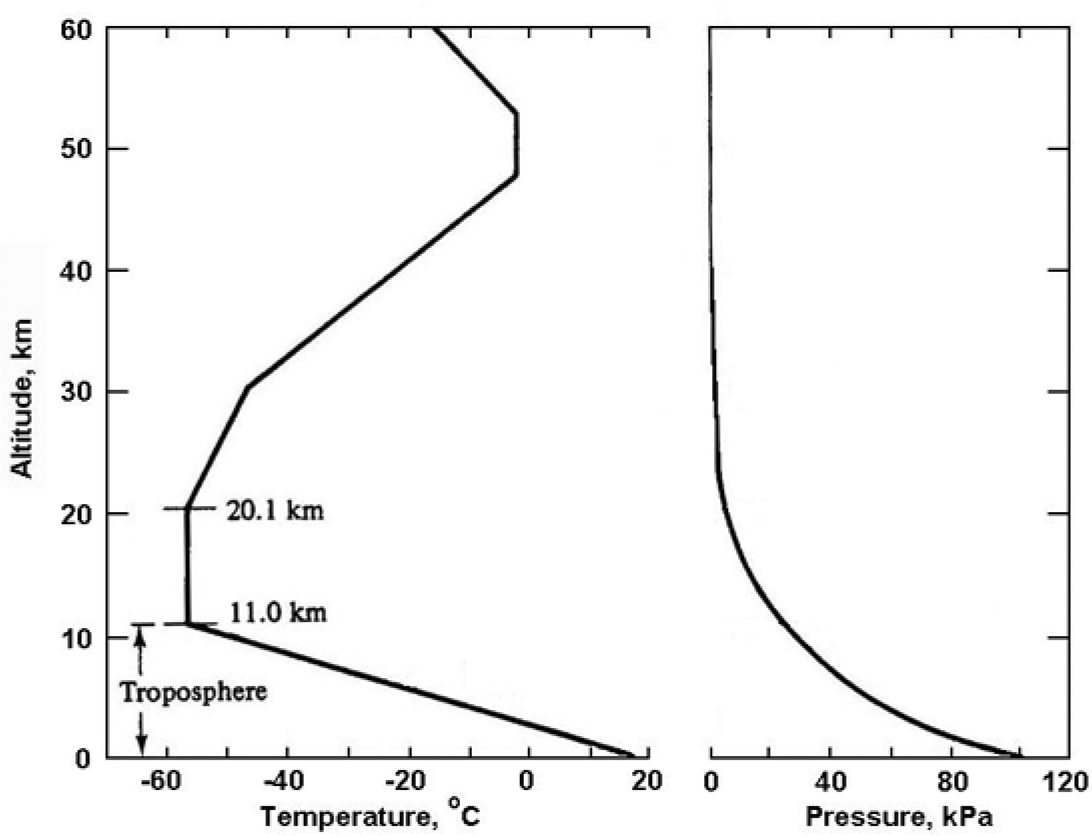

Figure 1: The U.S. standard atmosphere showing the variations in temperature and pressure with altitude.

As an example we will focus on the troposphere ($0 < y < 12,000\ m$) where, as illustrated in Figure 1, the temperature is approximately linear with altitude according to

$$T(y) = (288 - \beta y) \quad (\text{in } ^\circ K) \tag{Ce3}$$

where the constant β is $6.5 \times 10^{-3}\ ^\circ K/m$. Substituting this into the equation for dp/dy and rearranging it follows that

$$\frac{dp}{p} = -\frac{g dy}{\mathcal{R}(288 - \beta y)} \tag{Ce4}$$

and integrating

$$\ln p = \frac{g}{\beta \mathcal{R}} \ln (288 - \beta y) + \text{constant} \tag{Ce5}$$

Finally establishing the boundary condition that the pressure at the surface $y = 0$ is denoted by p_0 this can be written as

$$\frac{p}{p_0} = \left(1 - \frac{\beta y}{288}\right)^{\frac{g}{\beta \mathcal{R}}} \tag{Ce6}$$

This pressure distribution in the troposphere is also tabulated in Table 1. Note, for example, at an altitude of $10,000\ m$ (just above the summit of Everest) the pressure is $26.5\ kPa$ or roughly one quarter of the pressure at sea level.

A second possible way to proceed is to assume that the mixing between the layers of the atmosphere is sufficiently rapid that the gas does not have time to gain or lose heat as it moves from one altitude to another. Then the changes in pressure and density will be **adiabatic** and it is appropriate to set

$$p(y) = C(\rho(y))^\gamma \tag{Ce7}$$

where C is a proportionality constant and γ is the ratio of specific heats of the gas or gas mixture. The constant C is determined from the pressure and density at the surface, p_s and ρ_s. Substitution into equation (Ce1) leads to the differential equation

$$\frac{dp}{p^{\frac{1}{\gamma}}} = -\frac{gdy}{C^{\frac{1}{\gamma}}} \tag{Ce8}$$

and integrating from the surface $y = 0$ up to some altitude y we have

$$\int_{p_s}^{p} \frac{dp}{p^{\frac{1}{\gamma}}} = -\int_{0}^{y} \frac{gdy}{C^{\frac{1}{\gamma}}} \tag{Ce9}$$

which yields the following expression for the pressure as a function of altitude:

$$p = \left[\frac{(1-\gamma)}{\gamma} \frac{gy}{C^{\frac{1}{\gamma}}} + (p_s)^{\frac{(1-\gamma)}{\gamma}} \right]^{\frac{\gamma}{(1-\gamma)}} \tag{Ce10}$$

Thus the pressure decreases with altitude since $\gamma > 1$. Note that for small altitudes, y, this can be expanded to yield the expected result

$$p = p_s - \rho_s gy + O(y^2) \tag{Ce11}$$

The pressure and density of the atmosphere at the surface of the planet Venus are respectively 9260 kPa and 63 kg/m^3 and the acceleration due to gravity is 8.7 m/s^2. Up to an altitude of about 40 km the atmosphere is approximately adiabatic with a γ of about 1.2. The above equation (Ce10) then yields a pressure p at an altitude of 30 km of 1120 kPa or about one eighth the value at the surface. In contrast the linear term in equation (Ce11) would yield a negative pressure at this altitude.

EXERCISES

Problem Number	Subject	Solution
101A	On the pressure in a molten earth	Solution
101B	On the adiabatic atmosphere of Venus	Solution
101C	On the pressure in a fluid planet	Solution