The Connection between Drag and the Wake

An important result which can be derived using the momentum theorem (and represents a good example illustrating the value of the momentum theorem) is the connection between the drag on a finite object in a uniform stream and the wake which that object produces. Consider the object shown in black outline in the figure below. The outline of the wake, defined below, is shown in blue. Surrounding the object

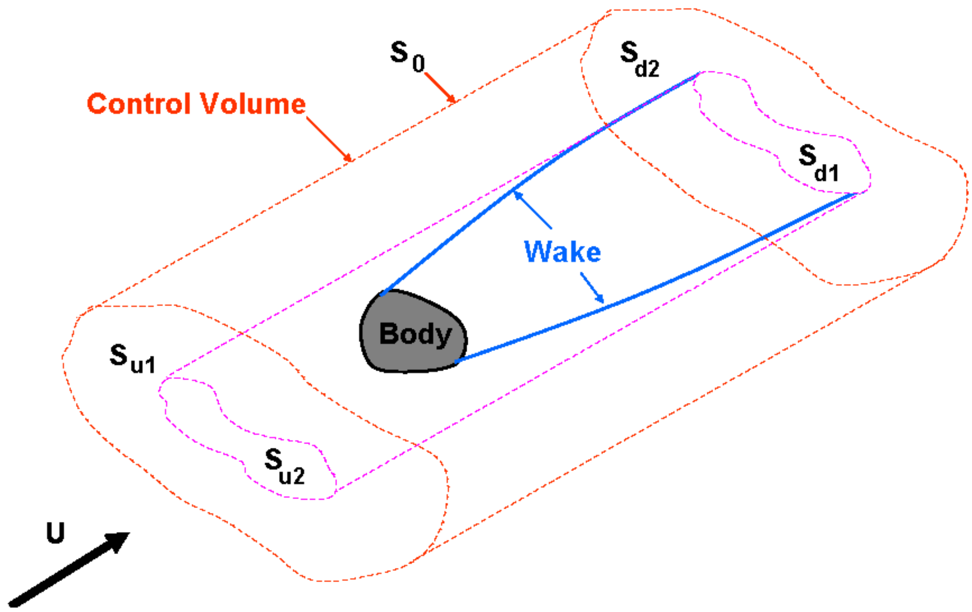

Figure 1: Definition of the control surfaces surrounding an object and its wake.

and its wake we draw a large cylindrical control volume as shown by the dashed red lines. The various components of the surface of this control volume are labeled S_{u1}, S_{u2}, S_{d1}, S_{d2}, and S_0. The surfaces S_{u1} and S_{u2} are flat and coplanar as are S_{d1} and S_{d2}; all four are normal to the oncoming uniform stream, U, and we will denote the velocity normal to these surfaces by u. On the other hand S_0 is everywhere parallel with U and we will denote the velocity normal to that surface (and to U) by u_n.

All these boundaries of the control volume are assumed sufficiently distant from the body so that the pressure and density, ρ, on all these surfaces are equal to the upstream pressure and density in the uniform stream (see further comment below). They are also sufficiently far away so that the u velocity on all these surfaces *except* S_{d2} is equal to U. Indeed this is what defines the boundary of the wake, namely that within the component of the surface area S_{d2}, the velocity u may be different from U. Then for convenience we also define the surface element S_{u2} as the projection of S_{d2} onto the upstream surface of the control volume as shown by the dashed mauve lines.

With these definitions we are ready to apply the continuity relation and the momentum theorem in the U direction. Assuming that the flow is steady so that the mass of fluid inside the control volume is unchanging then conservation of mass requires that

$$\int_{S_{u1}+S_{u2}} \rho u dS - \int_{S_{d1}+S_{d2}} \rho u dS - \int_{S_0} \rho u_n dS \;=\; 0 \qquad \text{(Bee1)}$$

Since $u = U$ on S_{u1}, S_{u2} and S_{d1} it follows that the integrals over S_{u1} and S_{d1} cancel and this relation reduces to

$$\int_{S_{d2}} \rho(U-u)dS = \int_{S_0} \rho u_n dS \tag{Bee2}$$

Now we apply the momentum theorem in the U direction to obtain the total force F acting on the contents of the control volume (which includes the body and the wake) in the U direction:

$$F = -\int_{S_{u1}+S_{u2}} \rho u^2 dS + \int_{S_{d1}+S_{d2}} \rho u^2 dS + \int_{S_0} \rho U u_n dS \tag{Bee3}$$

As with the continuity equation, since $u = U$ on S_{u1}, S_{u2} and S_{d1} the integrals over S_{u1} and S_{d1} cancel and the above equation reduces to

$$F = \int_{S_{d2}} \rho(u^2 - U^2)dS + U\int_{S_0} \rho u_n dS \tag{Bee4}$$

where the U in the second term is uniform and can therefore be taken outside of the integral. By substituting for the integral in the second term from the result obtained from the continuity equation we obtain

$$F = \int_{S_{d2}} \rho(u^2 - U^2)dS + U\int_{S_{d2}} \rho(U-u)dS \tag{Bee5}$$

or

$$F = -\int_{S_{d2}} \rho u(U-u)dS \tag{Bee6}$$

Finally we must consider the various possible contributions to the total force, F, acting on the control volume and its contents in the U direction. It is assumed that the flow has a sufficiently high Reynolds number so that the shear stresses acting on S_0 are negligible and so that there are no significant viscous contributions to the normal stresses on S_{u1}, S_{u2}, S_{d1}, and S_{d2}. Thus the only pertinent forces acting on the external surface of the control volume are those due to the pressure on S_{u1}, S_{u2}, S_{d1}, and S_{d2}. Moreover it is assumed that these surfaces are sufficiently far from the body that the pressure on all four surfaces is equal to the pressure in the uniform stream. It follows that there is no contribution of the pressures to F. Consequently if we neglect contributions from body forces such as gravity (or assume U is horizontal), the only contribution to F is the force that must be applied to the body to hold it in place against the drag force, D, imposed on the body by the flow. Consequently $F = -D$ and

$$D = \int_{S_{d2}} \rho u(U-u)dS \tag{Bee7}$$

This result clearly demonstrates the connection between the drag and the wake the body creates. If there were no wake so that $u = U$ within S_{d2} then the drag is zero. Also the drag is greater the larger the wake or the larger the "velocity defect in the wake", $(U-u)$.

It follows that a passive object which is being held in a wind or water tunnel by a strut support will produce a wake such as that shown in figure 2 where knowledge of the velocity profile in the wake would allow evaluation of the integral in the above equation and therefore measurement of the drag. Indeed this is one technique which has been used to measure the drag in such experiments.

Finally we note that if the body is self-propelled object such as an airplane or a submarine it must necessarily follow that the net force, F, on that object must be zero. It follows that the integral in the above equation must be zero and therefore the form of the velocity profile in the wake of a self-propelled

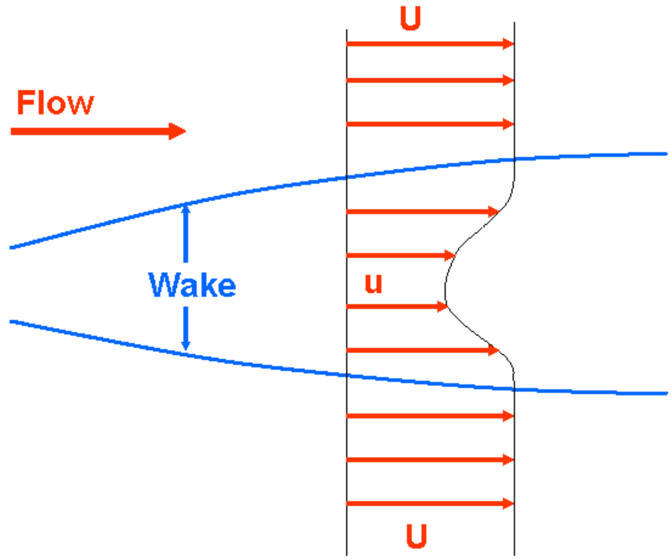

Figure 2: Typical velocity profile in a high Reynolds number wake.

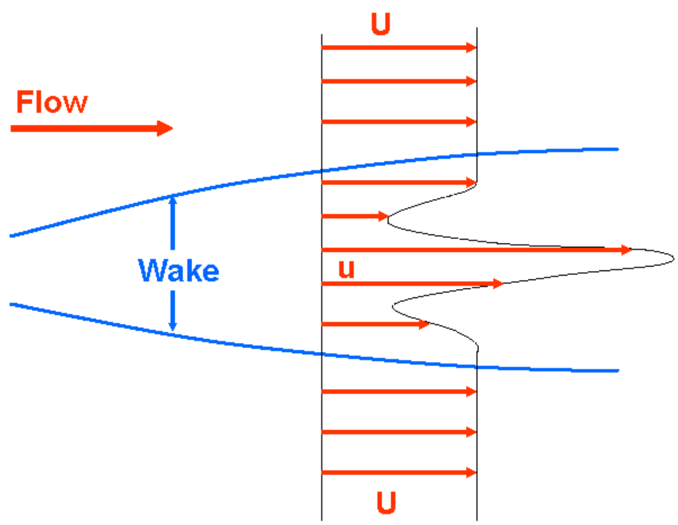

Figure 3: Typical velocity profile in the momentumless wake behind a self-propelled body.

object must reflect this conclusion. Such wakes in which the integral of $u(U-u)$ is zero are called "momentumless wakes" and must have regions in which $u > U$ as well as regions in which $u < U$ in order for the integral to be zero. An example is shown in figure 3 where one could visualize the region in the center of the wake in which $u > U$ as resulting from a single propeller behind a submarine while the hull produces the regions in which $u < U$ surrounding the propeller wake.