# Fish Locomotion

Some of the basic terminology used in referring to fish or cetacean anatomy is shown in Figure 1 though we will be primarily concerned in the present analysis with the tail (the caudal fin or fluke). As we shall see this hydrofoil structure provides a very efficient means of propulsion for the fish or whale and therefore the geometry of the caudal fin or fluke is very important in determining the effectiveness and efficiency of propulsion. Animals (like *bonita*) that rely heavily on their acceleration and speed through the water tend to have high aspect ratio tails of large span relative to the chord as exemplified in Figure 2 (left). Because of structural limitations, whales cannot reach such large aspect ratios (Figure 2 (right)).

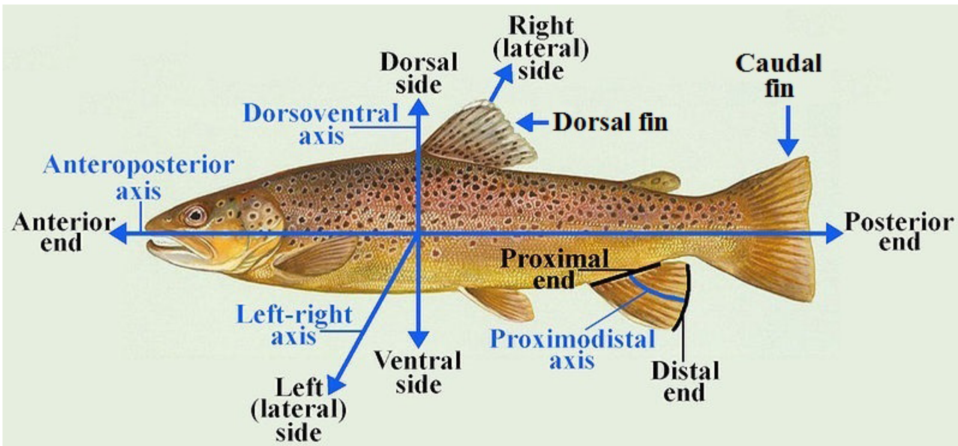


Figure 1: Fish fin nomenclature.

In a frame of reference fixed in a whale, the motion of the tail (the "fluke"), depicted in Figure 3, can be described as follows:

1. An oncoming uniform stream of velocity, $U$, representing the forward speed of the whale (in the direction, $x$).
2. A "heave" motion of the fluke in the direction, $y$, perpendicular to the $x$ direction. It is assumed that this motion, in combination with the forward motion, $U$, produces a sinusoidal trajectory of the

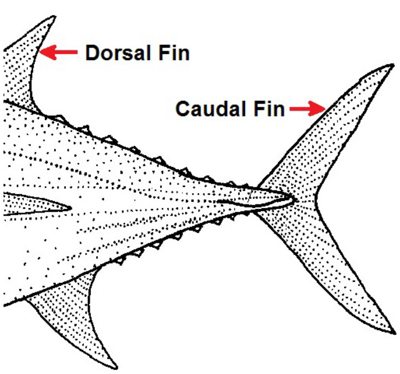


Figure 2: Left: High aspect ratio fins. Right: Whale fluke.

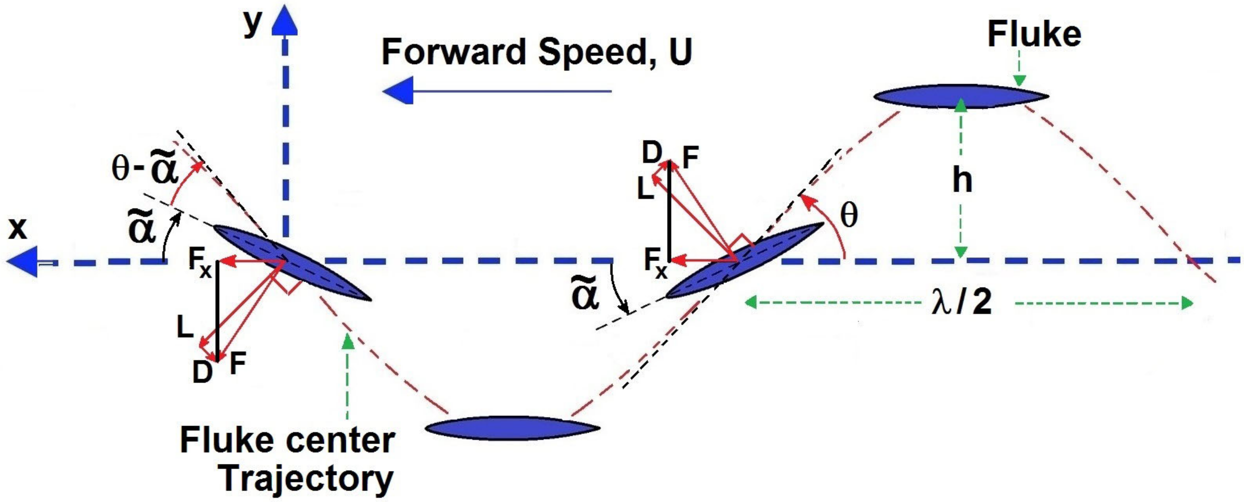


Figure 3: Caudal fin or fluke trajectory and orientation.

center of the fluke (as sketched in Figure 3) such that the center of the fluke moves according to

$$y \;=\; h \sin \omega t \tag{Dfd1}$$

where $\omega$ is the frequency of the oscillatory motion and $h$ is the amplitude. Consequently the wavelength. $\lambda$, of the motion is given by $\lambda = 2\pi U/\omega$. It also follows that the inclination of the trajectory with respect to the $x$ axis (as shown in Figure 3) is $\theta$ where

$$\tan \theta \;=\; \frac{dy}{dx} \;=\; \frac{1}{U}\frac{dy}{dt} \;=\; \frac{\omega h}{U} \cos \omega t \tag{Dfd2}$$

so that

$$\cos \theta \;=\; \left[1 + \frac{\omega^2 h^2}{U^2} \cos^2 \omega t \right]^{-\frac{1}{2}} \qquad \text{and} \qquad \sin \theta \;=\; \frac{\omega h}{U} \cos \omega t \left[1 + \frac{\omega^2 h^2}{U^2} \cos^2 \omega t \right]^{-\frac{1}{2}} \tag{Dfd3}$$

3. A oscillatory rotation of the fluke about its center (or pitching motion) such that the inclination of the fluke relative to the $x$ direction is $\alpha$ where

$$\alpha \;=\; \tilde{\alpha} \cos \omega t \tag{Dfd4}$$

We note that the phase relationship of the second and third of these oscillatory motions is such that the relative pitching motion, $\theta - \alpha$, lags the heaving motion by $\pi/2$. This common feature of fish, whale and bird motion is mostly achieved passively by the nature of the structure of the spine, fluke or wing though, as we will see in the section on bird flight, the complex motions in that case require additional active features. In the case of fish and cetaceans considerable musculature is needed to perform the heaving motion and the pitching motion mostly results from the reaction of the structure to that heaving motion. As a side note we observe that the basic principles described in this section were used by a California engineer, Calvin Gongwer, to invent the swimming machine shown in Figure 4 that results in remarkably fast and efficient human propulsion underwater. The device is strapped to the lower legs at one end and flexing at the knees produces the heaving motion of the high aspect ratio foils located below the chest and at the ankles (the pitching motion is passively controlled by a spring system that adjusts the pitch angle during the heave). The author can personally attest to the remarkable performance of this device.

Figure 4: Michael Wilson with the swimming machine invented by Cal Gongwer and used by the US Navy. (California Institute of Technology, 1969.)

Gongwer sold the design to the US Navy by easily outpacing their frogmen on a 20 mile plus swim from the California coast to Catalina Island. We should also note that common swim fins operate on the same principle by with much less effectiveness and efficiency because of their much smaller aspect ratio.

Returning to the above analysis of fish and cetacea propulsion, it follows that the angle of incidence of the fluke with respect to its *relative* motion through the fluid, $\alpha^*$, is given by

$$\alpha^* \;=\; \theta - \alpha \;= \arctan\left\{\frac{\omega h}{U}\cos\omega t\right\} - \tilde{\alpha}\cos\omega t \qquad \text{(Dfd5)}$$

Moreover the magnitude of the velocity of relative motion, $V$, is given by

$$V^2 \;=\; U^2 + (\frac{dy}{dt})^2 \;=\; U^2 + \omega^2 h^2 \cos^2(\omega t) \qquad \text{(Dfd6)}$$

and therefore the lift and drag forces acting on the fluke (planform area, $A_F$), $L$ and $D$, are given by $\frac{1}{2}\rho V^2 A_F C_L$ and $\frac{1}{2}\rho V^2 A_F C_D$ respectively where $C_L$ and $C_D$ are the lift and drag coefficients for the fluke. It follows that the component of the instantaneous propulsive force in the $x$ direction, $F_x$, is given by

$$\frac{F_x}{\frac{1}{2}\rho U^2 A_F} \;=\; \left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}\{C_L\sin\theta - C_D\cos\theta\} \;=\; \left\{C_L(\frac{\omega h}{U})\cos\omega t - C_D\right\}\left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}} \qquad \text{(Dfd7)}$$

The force in the $y$ direction normal to the forward motion, $F_y$, is given by

$$\frac{F_y}{\frac{1}{2}\rho U^2 A_F} \;=\; \left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}\{C_L\cos\theta + C_D\sin\theta\} \;=\; \left\{C_L + C_D(\frac{\omega h}{U})\cos\omega t\right\}\left\{1 + \frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}} \qquad \text{(Dfd8)}$$

Clearly by symmetry the average of $F_y$ over one cycle is zero and this is appropriate for a neutrally buoyant fish. However, a non-neutrally buoyant fish might require a non-zero lateral force in order to balance its weight or buoyancy. This can be achieved by adding in an additional constant angle of incidence as we will do in the context of bird flight in section (Dfe).

The thrust, $T$, is given by the mean value of $F_x$ averaged over one cycle of the oscillation of the fluke and therefore

$$T^* \;=\; \frac{T}{\frac{1}{2}\rho U^2 A_F} \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left\{C_L(\frac{\omega h}{U})\cos\omega t - C_D\right\}\left\{1+\frac{\omega^2 h^2}{U^2}\cos^2\omega t\right\}^{\frac{1}{2}}\, d(\omega t) \qquad \text{(Dfd9)}$$

Before going any further it is evident that the thrust produced by the motion of the fluke is due to the component of the lift force in the $x$ direction which is positive during the entire fluke cycle (except at $y = \pm\pi/2$ where it becomes zero). It is also clear that the drag, $D$, on the fluke will decrease that propulsive force and so we recognize again the importance of (1) a large lift/drag ratio, $L/D$, (2) a well-designed fluke cross-section and (3) as large an aspect ratio as physically possible. All the very fastest fish and cetaceans have fins with a large aspect ratio, that is with a large span and small chord.

To evaluate the integral in equation (Dfd8) it remains to specify $C_L$ and $C_D$ which, if we assume that the oscillatory motions are sufficiently slow that quasisteady coefficients can be used, will be functions primarily of the local angle of incidence, $\alpha^* = \theta - \alpha$. For the purposes of this demonstration we will neglect the drag and assume the lift coefficient is the same as that of a flat plate namely

$$C_L \approx 2\pi\sin\alpha^* \;=\; 2\pi\sin\{\theta - \tilde{\alpha}\cos\omega t\} \qquad \text{(Dfd10)}$$

which, using the expressions (Dfd3), becomes

$$C_L \;=\; 2\pi\left[\frac{\omega h}{U}\cos\omega t\cos\{\tilde{\alpha}\cos\omega t\} - \sin\{\tilde{\alpha}\cos\omega t\}\right]\left[1+\frac{\omega^2 h^2}{U^2}\cos^2\omega t\right]^{-\frac{1}{2}} \qquad \text{(Dfd11)}$$

Substituting this into equation (Dfd8) and setting $C_D = 0$ yields

$$T^* = \frac{T}{\frac{1}{2}\rho U^2 A_F} = \frac{\omega h}{U}\int_0^{2\pi}\left[\frac{\omega h}{U}\cos\omega t\cos\{\tilde{\alpha}\cos\omega t\} - \sin\{\tilde{\alpha}\cos\omega t\}\right]\cos\omega t\; d(\omega t) \qquad \text{(Dfd12)}$$

Note that the dimensionless thrust, $T^*$, is a function only of the two dimensionless parameters, $\omega h/U$and $\tilde{\alpha}$. If the angle $\tilde{\alpha}$ is small the the above expression may be approximately evaluated as

$$T^* \;\approx\; \pi\frac{\omega h}{U}\left\{\frac{\omega h}{U} - \tilde{\alpha}\right\} \qquad \text{(Dfd13)}$$

Note that $T$ is zero when the transient effective angle of incidence, $\theta - \tilde{\alpha}$, is zero and the foil follows the sinusoidal path without any inclination to it. Note also that, perhaps surprisingly, the thrust takes the substantial but characteristic value of $\pi\omega^2 h^2/U^2$ when $\tilde{\alpha} = 0$ and the heave motion occurs with zero inclination of the foil to the direction of motion. Also note that a negative value of $\tilde{\alpha}$ yields an increased thrust though there will be a limit to this when the maximum transient angle of incidence, $\theta - \alpha$, exceeds the value at which the foil stalls.

The thrust, $T$, will be balanced by the drag on the entire fish (or cetacean), $D_B$, so that

$$T \;=\; D_B \;=\; \frac{1}{2}\rho A_B U^2 C_{DB} \qquad \text{(Dfd14)}$$

where $C_{DB}$ is the drag coefficient for the entire fish based on an area, $A_B$, which might be conveniently chosen as the maximum cross-sectional area of the fish normal to the direction of motion. It follows that the swimming velocity of the animal, $U$, is given by the solution of $T^* = C_{DB}A_B/A_F$ and this yields

$$U \;=\; \frac{\pi\omega h A_F}{2A_B C_{DB}}\left[\left\{\tilde{\alpha}^2 + \frac{4A_B C_{DB}}{\pi A_f}\right\}^{\frac{1}{2}} - \tilde{\alpha}\right] \qquad \text{(Dfd15)}$$

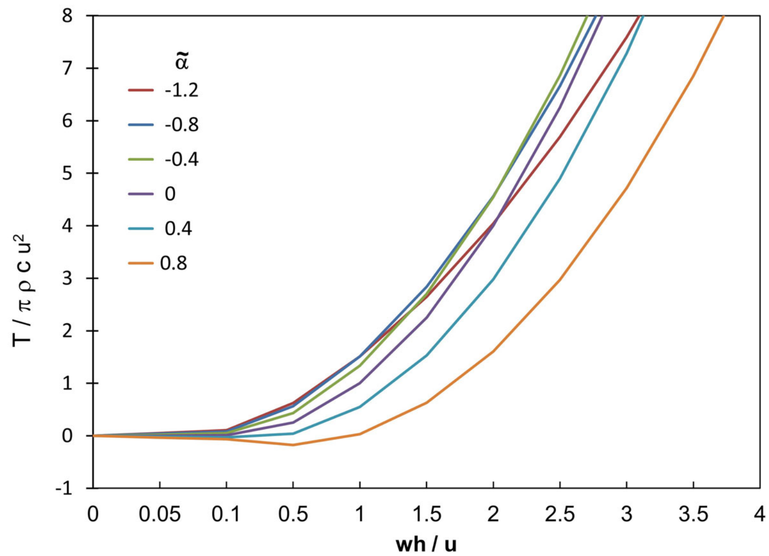


Figure 5: Variation of the dimensionless thrust, $\overline{T}/\pi\rho c U^2$, with dimensionless frequency, $\omega h/U$ for various $\tilde{\alpha}$ (in radians).

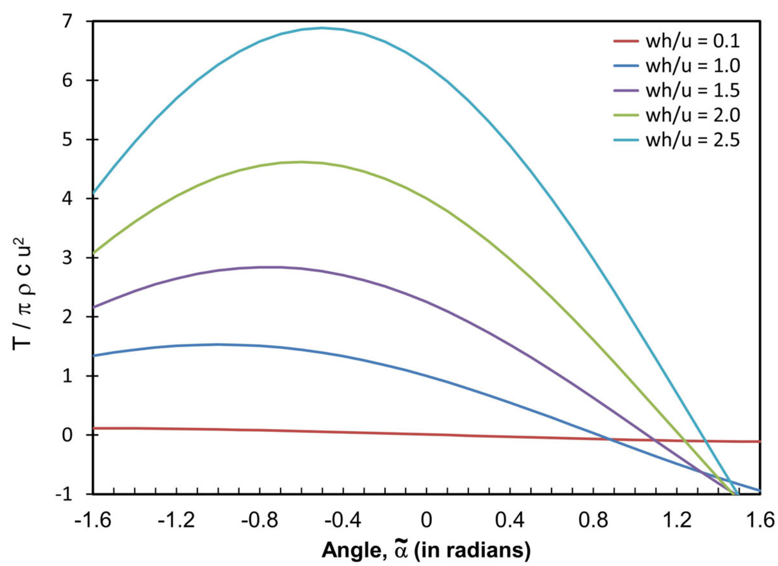


Figure 6: Variation of the dimensionless thrust, $\overline{T}/\pi\rho c U^2$, with the angle, $\tilde{\alpha}$ (in radians).

and, for the characteristic motion in which $\tilde{\alpha} = 0$ this yields

$$U \;=\; \omega h \left\{ \frac{\pi A_F}{A_B C_{DB}} \right\}^{\frac{1}{2}} \tag{Dfd16}$$