Method of Characteristics

The typical numerical solution by the method of characteristics is depicted graphically in figure 1. The time interval, δt, and the spatial increment, δs, are specified. Then, given all values of the two dependent variables (say u and h^*) at one instant in time, one proceeds as follows to find all the values at points such as C at a time δt later. The intersection points, A and B, of the characteristics through C are first determined. Then interpolation between the known values at points such as R, S and T are used to determine the values of the dependent variables at A and B. The values at C follow from equations such as (Bnfb15) and (Bnfb16) or some alternative version. Repeating this for all points at time $t + \delta t$ allows one to march forward in time.

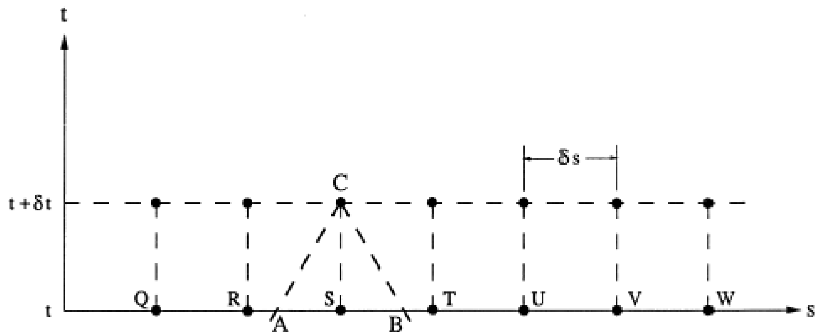

Figure 1: Example of numerical solution by method of characteristics.

There is, however, a maximum time interval, δt, that will lead to a stable numerical solution. Typically this requires that δt be less than $\delta x/c$. In other words, it requires that the points A and B of figure 1 lie *inside* of the interval RST. The reason for this condition can be demonstrated in the following way. Assume for the sake of simplicity that the slopes of the characteristics are $\pm c$; then the distances $AS = SB = c\delta t$. Using linear interpolation to find u_A and u_B from u_R, u_S and u_T leads to

$$u_A = u_S \left(1 - c\frac{\delta t}{\delta s}\right) + u_R \frac{c\delta t}{\delta s} \tag{Bnfc1}$$

$$u_B = u_S \left(1 - c\frac{\delta t}{\delta s}\right) + u_T \frac{c\delta t}{\delta s} \tag{Bnfc2}$$

Consequently, an error in u_R of, say, δu would lead to an error in u_A of $c\delta u\delta t/\delta s$ (and similarly for u_T and u_B). Thus the error would be magnified with each time step unless $c\delta t/\delta s < 1$ and, therefore, the numerical integration is only stable if $\delta t < \delta x/c$. In many hydraulic system analyses this places a quite severe restriction on the time interval δt, and often necessitates a large number of time steps.

A procedure like the above will also require boundary conditions to be specified at any mesh point which lies either, at the end of a pipe or, at a junction of the pipe with a pipe of different size (or a pump or any other component). If the points S and C in figure 1 were end points, then only one characteristic would lie within the pipe and only one relation, (Bnfb13) or (Bnfb14), can be used. Therefore, the boundary condition must provide a second relation involving u_C or h^*_C (or both). An example is an open-ended pipe for which the pressure and, therefore, h^* is known. Alternatively, at a junction between two sizes of pipe, the two required relations will come from one characteristic in each of the two pipes, plus a continuity equation at the junction ensuring that the values of uA_0 in both pipes are the same at the junction. For this reason it is sometimes convenient to rewrite equations (Bnfb11) and (Bnfb12) in terms of the volume flow rate $Q = uA_0$ instead of u so that

1. On $\frac{ds}{dt} = u + c$

$$\frac{dQ}{dt} + \frac{A_0 g}{c}\frac{dh^*}{dt} + \frac{Qc}{A_0}\frac{dA_0}{ds} - \frac{Qg_s}{c} + \frac{f}{4aA_0}Q|Q| = 0 \tag{Bnfc3}$$

2. On $\frac{ds}{dt} = u - c$

$$\frac{dQ}{dt} - \frac{A_0 g}{c}\frac{dh^*}{dt} - \frac{Qc}{A_0}\frac{dA_0}{ds} + \frac{Qg_s}{c} + \frac{f}{4aA_0}Q|Q| = 0 \tag{Bnfc4}$$

Even in simple pipe flow, additional complications arise when the instantaneous pressure falls below vapor pressure and cavitation occurs. In the context of water-hammer analysis, this is known as "water column separation", and is of particular concern because the violent collapse of the cavity can cause severe structural damage (see, for example, Martin 1978). Furthermore, the occurrence of water column separation can trigger a series of cavity formations and collapses, resulting in a series of impulsive loads on the structure. The possibility of water column separation can be tracked by following the instantaneous pressure. To proceed beyond this point requires a procedure to incorporate a cavity in the waterhammer calculation using the method of characteristics. A number of authors (for example, Tanahashi and Kasahara 1969, Weyler *et al.* 1971, Safwat and van der Polder 1973) have shown that this is possible. However the calculated results after the first collapse can deviate substantially from the observations. This is probably due to the fact that the first cavity is often a single, coherent void. This will shatter into a cloud of smaller bubbles as a result of the violence of the first collapse. Subsequently, one is dealing with a bubbly medium whose wave propagation speeds may differ significantly from the acoustic speed assumed in the analytical model. Other studies have shown that qualitatively similar changes in the water-hammer behavior occur when gas bubbles form in the liquid as a result of dissolved gas coming out of solution (see, for example, Wiggert and Sundquist 1979).

In many time domain analyses, turbomachines are treated by assuming that the temporal rates of change are sufficiently slow that the turbomachine responds quasistatically, moving from one steady state operating point to another. Consequently, if points A and B lie at inlet to and discharge from the turbomachine then the equations relating the values at A and B would be

$$Q_B = Q_A = Q \tag{Bnfc5}$$

$$h_B^* = h_A^* + H(Q) \tag{Bnfc6}$$

where $H(Q)$ is the head rise across the machine at the flow rate, Q. Data presented later will show that the quasistatic assumption is only valid for rates of change less than about one-tenth the frequency of shaft rotation. For frequencies greater than this, the pump dynamics become important (see sections Bngi, Bngj).

For more detailed accounts of the methods of characteristics the reader is referred to Streeter and Wylie (1967), or any modern text on numerical methods. Furthermore, there are a number of standard codes available for time domain analysis of transients in hydraulic systems, such as that developed by Amies, Levek and Struesseld (1977). The methods work well so long as one has confidence in the differential equations and models which are used. In other circumstances, such as occur in two-phase flow, in cavitating flow, or in the complicated geometry of a turbomachine, the time domain methods may be less useful than the alternative frequency domain methods to which we now turn.