# Properties of Transfer Matrices

Transfer matrices (and transmission matrices) have some fundamental properties that are valuable to recall when constructing or evaluating the dynamic properties of a component or system.

We first identify a "uniform" distributed component as one in which the differential equations (for example, equation (Bngc8)) governing the fluid motion have coefficients which are independent of position, $s$. Then, for the class of systems represented by the equation (Bngc8), the matrix $[F]$ is independent of $s$. For a system of order two, the transfer function $[T]$ would take the explicit form given by equations (Bngd3).

To determine another property of this class of dynamic systems, consider that the equations (Bngc8) have been manipulated to eliminate all but one of the unknown fluctuating quantities, say $\tilde{q}^1$. The resulting equation will take the form

$$\sum_{n=0}^{N} a_n(s) \frac{d^n \tilde{q}^1}{ds^n} = 0 \qquad \text{(Bngf1)}$$

In general, the coefficients $a_n(s)$, $n = 0 \rightarrow N$, will be complex functions of the mean flow and of the frequency. It follows that there are $N$ independent solutions which, for all the independent fluctuating quantities, may be expressed in the form

$$\{\tilde{q}^n\} = [B(s)]\,\{A\} \qquad \text{(Bngf2)}$$

where $[B(s)]$ is a matrix of complex solutions and $\{A\}$ is a vector of arbitrary complex constants to be determined from the boundary conditions. Consequently, the inlet and discharge fluctuations denoted by subscripts 1 and 2, respectively, are given by

$$\{\tilde{q}_1^n\} = [B(s_1)]\,\{A\} \quad ; \quad \{\tilde{q}_2^n\} = [B(s_2)]\,\{A\} \qquad \text{(Bngf3)}$$

and therefore the transfer function

$$[T] = [B(s_2)]\,[B(s_1)]^{-1} \qquad \text{(Bngf4)}$$

Now for a uniform system, the coefficients $a_n$ and the matrix $[B]$ are independent of $s$. Hence the equation (Bngf1) has a solution of the form

$$[B(s)] = [C]\,[E] \qquad \text{(Bngf5)}$$

where $[C]$ is a known matrix of constants, and $[E]$ is a diagonal matrix in which

$$E_{nn} = e^{j\gamma_n s} \qquad \text{(Bngf6)}$$

where $\gamma_n,\ n = 1$ to $N$, are the solutions of the dispersion relation

$$\sum_{n=0}^{N} a_n \gamma^n = 0 \qquad \text{(Bngf7)}$$

Note that $\gamma_n$ are the wavenumbers for the $N$ types of wave of frequency, $\omega$, which can propagate through the uniform system. In general, each of these waves has a distinct wave speed, $c_n$, given by $c_n = -\omega/\gamma_n$. It follows from equations (Bngf5), (Bngf6) and (Bngf4) that the transfer matrix for a uniform distributed system must take the form

$$[T] = [C]\,[E^*]\,[C]^{-1} \qquad \text{(Bngf8)}$$

where $[E^*]$ is a diagonal matrix with

$$E^*_{nn} = e^{j\gamma_n \ell} \tag{Bngf9}$$

and $\ell = s_2 - s_1$.

An important diagnostic property arises from the form of the transfer matrix, (Bngf8), for a uniform distributed system. The determinant, $D_T$, of the transfer matrix $[T]$ is

$$D_T = \exp\left\{j\left(\gamma_1 + \gamma_2 + \cdots + \gamma_N\right)\ell\right\} \tag{Bngf10}$$

Thus the value of the determinant is related to the sum of the wavenumbers of the $N$ different waves which can propagate through the uniform distributed system. Furthermore, if all the wavenumbers, $\gamma_n$, are purely real, then

$$|D_T| = 1 \tag{Bngf11}$$

The property that the modulus of the determinant of the transfer function is unity will be termed "quasi-reciprocity" and will be discussed further below. Note that this will only be the case in the absence of wave damping when $\gamma_n$ and $c_n$ are purely real.

Turning now to another property, a system is said to be "reciprocal" if, in the matrix $[Z]$ defined by

$$\left\{\begin{matrix} \tilde{p}_1^T \\ \tilde{p}_2^T \end{matrix}\right\} = [Z] \left\{\begin{matrix} \tilde{m}_1 \\ -\tilde{m}_2 \end{matrix}\right\} \tag{Bngf12}$$

the transfer impedances $Z_{12}$ and $Z_{21}$ are identical (see Brown 1967 for the generalization of this property in systems of higher order). This is identical to the condition that the determinant, $D_T$, of the transfer matrix $[T]$ be unity:

$$D_T = 1 \tag{Bngf13}$$

We shall see that a number of commonly used components have transfer functions which are reciprocal. In order to broaden the perspective we have introduced the property of "quasi-reciprocity" to signify those components in which the modulus of the determinant is unity or

$$|D_T| = 1 \tag{Bngf14}$$

We have already noted that uniform distributed components with purely real wavenumbers are quasi-reciprocal. Note that a uniform distributed component will only be reciprocal when the wavenumbers tend to zero, as, for example, in incompressible flows in which the wave propagation speeds tend to infinity.

By utilizing the results of the section on combinations of transfer matrices, we can conclude that any series or parallel combination of reciprocal components will yield a reciprocal system. Also a series combination of quasi-reciprocal components will be quasi-reciprocal. However it is *not* necessarily true that a parallel combination of quasi-reciprocal components is quasi-reciprocal.

An even more restrictive property than reciprocity is the property of "symmetry". A "symmetric" component is one that has identical dynamical properties when turned around so that the discharge becomes the inlet, and the directional convention of the flow variables is reversed (Brown 1967). Then, in contrast to the regular transfer matrix, $[T]$, the effective transfer matrix under these reversed circumstances is $[TR]$ where

$$\left\{\begin{matrix} \tilde{p}_1^T \\ -\tilde{m}_1 \end{matrix}\right\} = [TR] \left\{\begin{matrix} \tilde{p}_2^T \\ -\tilde{m}_2 \end{matrix}\right\} \tag{Bngf15}$$

and, comparing this with the definition (Bngc2), we observe that

$$TR_{11} = T_{22}/D_T \quad ; \quad TR_{12} = T_{12}/D_T$$

$$TR_{21} = T_{21}/D_T \quad ; \quad TR_{22} = T_{11}/D_T \tag{Bngf16}$$

Therefore symmetry, $[T] = [TR]$, requires

$$T_{11} = T_{22} \qquad \text{and} \qquad D_T = 1 \tag{Bngf17}$$

Consequently, in addition to the condition, $D_T = 1$, required for reciprocity, symmetry requires $T_{11} = T_{22}$.

As with the properties of reciprocity and quasi-reciprocity, it is useful to consider the property of a system comprised of symmetric components. Note that according to the combination rules, a parallel combination of symmetric components is symmetric, whereas a series combination may not retain this property. In this regard symmetry is in contrast to quasi-reciprocity in which the reverse is true.

In the case of uniform distributed systems, Brown (1967) shows that symmetry requires

$$F_{11} = F_{22} = 0 \tag{Bngf18}$$

so that the solution of the equation (Bngd4) for $\lambda$ is $\lambda = \pm\lambda^*$ where $\lambda^* = (F_{21}F_{12})^{\frac{1}{2}}$ is known as the "propagation operator" and the transfer function (Bngd3) becomes

$$T_{11} = T_{22} = \cosh \lambda^* \ell$$

$$T_{12} = Z_C \sinh \lambda^* \ell$$

$$T_{21} = Z_C^{-1} \sinh \lambda^* \ell \tag{Bngf19}$$

where $Z_C = (F_{12}/F_{21})^{\frac{1}{2}} = (T_{12}/T_{21})^{\frac{1}{2}}$ is known as the "characteristic impedance".

In addition to the above properties of transfer functions, there are also properties associated with the net flux of fluctuation energy into the component or system. These will be elucidated after we have examined some typical transfer functions for components of hydraulic systems.