Fluctuation Energy Flux

It is clearly important to be able to establish the net energy flux into or out of a hydraulic system component (see Brennen and Braisted 1980). If the fluid is incompressible, and the order two system is characterized by the mass flow rate, m, and the total pressure, p^T, then the instantaneous energy flux through any system node is given by mp^T/ρ where the density is assumed constant. Substituting the expansions (Bnga2) and (Bnga3) for p^T and m, it is readily seen that the mean flux of energy due to the fluctuations, E, is given by

$$E = \frac{1}{4\rho}\left\{\tilde{m}\bar{\tilde{p}}^T + \bar{\tilde{m}}\tilde{p}^T\right\} \qquad \text{(Bngh1)}$$

where the overbar denotes a complex conjugate. Superimposed on E are fluctuations in the energy flux whose time-average value is zero, but we shall not be concerned with those fluctuations. The mean fluctuation energy flux, E, is of more consequence in terms, for example, of evaluating stability. It follows that the net flux of fluctuation energy *into* a component from the fluid is given by

$$E_1 - E_2 = \Delta E = \frac{1}{4\rho}\left[\tilde{m}_1\bar{\tilde{p}}_1^T + \bar{\tilde{m}}_1\tilde{p}_1^T - \tilde{m}_2\bar{\tilde{p}}_2^T - \bar{\tilde{m}}_2\tilde{p}_2^T\right] \qquad \text{(Bngh2)}$$

and when the transfer function form (Bngc2) is used to write this in terms of the inlet fluctuating quantities

$$\Delta E = \frac{1}{4\rho}\left[-\Gamma_1\tilde{p}_1^T\bar{\tilde{p}}_1^T - \Gamma_2\tilde{m}_1\bar{\tilde{m}}_1 + (1-\Gamma_3)\tilde{m}_1\bar{\tilde{p}}_1^T + (1-\bar{\Gamma}_3)\bar{\tilde{m}}_1\tilde{p}_1^T\right] \qquad \text{(Bngh3)}$$

where

$$\begin{aligned}\Gamma_1 &= T_{11}\bar{T}_{21} + T_{21}\bar{T}_{11}\\ \Gamma_2 &= T_{22}\bar{T}_{12} + T_{12}\bar{T}_{22}\\ \Gamma_3 &= \bar{T}_{11}T_{22} + \bar{T}_{21}T_{12}\end{aligned} \qquad \text{(Bngh4)}$$

and

$$|\Gamma_3|^2 = |D_T|^2 + \Gamma_1\Gamma_2 \qquad \text{(Bngh5)}$$

Using the above relations, we can draw the following conclusions:

1. A component or system which is "conservative" (in the sense that $\Delta E = 0$ under all circumstances, whatever the values of $\tilde{p}_1^T$ and $\tilde{m}_1$) requires that

$$\Gamma_1 = \Gamma_2 = 0 \quad , \quad \Gamma_3 = 1 \qquad \text{(Bngh6)}$$

and these in turn require not only that the system or component be "quasi-reciprocal" ($|D_T| = 1$) but also that

$$\frac{\bar{T}_{11}}{T_{11}} = -\frac{\bar{T}_{12}}{T_{12}} = -\frac{\bar{T}_{21}}{T_{21}} = \frac{\bar{T}_{22}}{T_{22}} = \frac{1}{D_T} \qquad \text{(Bngh7)}$$

Such conditions virtually never occur in real hydraulic systems, though any combination of lumped inertances and compliances does constitute a conservative system. This can be readily demonstrated as follows. An inertance or compliance has $D_T = 1$, purely real T_{11} and T_{22} so that $T_{11} = \bar{T}_{11}$ and $T_{22} = \bar{T}_{22}$, and purely imaginery T_{21} and T_{12} so that $T_{21} = -\bar{T}_{21}$ and $T_{12} = -\bar{T}_{12}$. Hence individual inertances or compliances satisfy equations (Bngh6) and (Bngh7). Furthermore, from the section "Combinations of Transfer Matrices", it can readily be seen that all combination of components with purely real T_{11} and T_{22} and purely imaginery T_{21} and T_{12} will retain the same properties. Consequently, any combination of inertance and compliance satisfies equations (Bngh6) and (Bngh7) and is conservative.

2. A component or system will be considered "completely passive" if ΔE is positive for all possible values of $\tilde{m}_1$ and $\tilde{p}_1^T$. This implies that a net external supply of energy to the fluid is required to maintain any steady state oscillation. To find the characteristics of the transfer function which imply "complete passivity" the expression (Bngh3) is rewritten in the form

$$\Delta E = \frac{|\tilde{p}_1^T|^2}{4\rho}\left[-\Gamma_1 - \Gamma_2 x\bar{x} + (1-\Gamma_3)x + (1-\bar{\Gamma}_3)\bar{x}\right] \tag{Bngh8}$$

where $x = \tilde{m}_1/\tilde{p}_1^T$. It follows that the sign of ΔE is determined by the sign of the expression in the square brackets. Moreover, if $\Gamma_2 < 0$, it is readily seen that this expression has a minimum and is positive for all x if

$$\Gamma_1\Gamma_2 > |1-\Gamma_3|^2 \tag{Bngh9}$$

which, since $\Gamma_2 < 0$, implies $\Gamma_1 < 0$. It follows that necessary and sufficient conditions for a component or system to be completely passive are

$$\Gamma_1 < 0 \quad \text{and} \quad G < 0 \tag{Bngh10}$$

where

$$G = |1-\Gamma_3|^2 - \Gamma_1\Gamma_2 = |D_T|^2 + 1 - 2Re\{\Gamma_3\} \tag{Bngh11}$$

The conditions (Bngh10) also imply $\Gamma_2 < 0$. Conversely a "completely active" component or system which always has $\Delta E < 0$ occurs if and only if $\Gamma_1 > 0$ and $G < 0$ which imply $\Gamma_2 > 0$. These properties are not, of course, the only possibilities. A component or system which is not completely passive or active could be "potentially active." That is to say, ΔE could be negative for the right combination of $\tilde{m}_1$ and $\tilde{p}_1^T$, which would, in turn, depend on the rest of the system to which the particular component or system is attached. Since Γ_1 is almost always negative, it transpires that most components are either completely passive or potentially active, depending on the sign of the quantity, G, which will therefore be termed the "dynamic activity". These circumstances can be presented graphically as shown in figure 1.

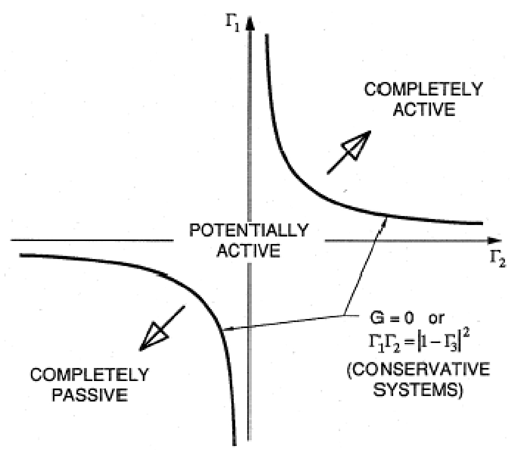

Figure 1: Schematic of the conditions for completely active, completely passive and potentially active components or systems.

In practice, of course, both the transfer function, and properties like the dynamic activity, G, will be functions not only of frequency but also of the mean flow conditions. Hence the potential for system instability should be evaluated by tracking the graph of G against frequency, and establishing the mean flow conditions for which the quantity G becomes negative within the range of frequencies for which transfer function information is available.

While the above analysis represents the most general approach to the stability of systems or components, the results are not readily interpreted in terms of commonly employed measures of the system or component characteristics. It is therefore instructive to consider two special subsets of the general case, not only because of the simplicity of the results, but also because of the ubiquity of these special cases. Consider first a system or component that discharges into a large, constant head reservoir, so that $\tilde{p}_2^T = 0$. It follows from the expression (Bngh2) that

$$\Delta E = \frac{|\tilde{m}_1|^2}{2\rho} Re\{\tilde{p}_1^T/\tilde{m}_1\} \qquad \text{(Bngh12)}$$

Note that ΔE is always purely real and that the sign only depends on the real part of the "input impedance"

$$\tilde{p}_1^T/\tilde{m}_1 = -T_{12}/T_{11} \qquad \text{(Bngh13)}$$

Consequently a component or system with a constant head discharge will be dynamically stable if the "input resistance" is positive or

$$Re\left\{-T_{12}/T_{11}\right\} > 0 \qquad \text{(Bngh14)}$$

This relation between the net fluctuation energy flux, the input resistance, and the system stability, is valuable because of the simplicity of its physical interpretation. In practice, the graph of input resistance against frequency can be monitored for changes with mean flow conditions. Instabilities will arise at frequencies for which the input resistance becomes negative.

The second special case is that in which the component or system begins with a constant head reservoir rather than discharging into one. Then

$$\Delta E = \frac{|\tilde{m}_2|^2}{2\rho} Re\left\{-\tilde{p}_2^T/\tilde{m}_2\right\} \qquad \text{(Bngh15)}$$

and the stability depends on the sign of the real part of the "discharge impedance"

$$-\tilde{p}_2^T/\tilde{m}_2 = -T_{12}/T_{22} \qquad \text{(Bngh16)}$$

Thus a constant head inlet component or system will be stable when the "discharge resistance" is positive or

$$Re\left\{-T_{12}/T_{22}\right\} > 0 \qquad \text{(Bngh17)}$$

In practice, since T_{11} and T_{22} are close to unity for many components and systems, both the condition (Bngh14) and the condition (Bngh17) reduce to the approximate condition that the system resistance, $Re\{-T_{12}\}$, be positive for system stability. While not always the case, this approximate condition is frequently more convenient and more readily evaluated than the more precise conditions detailed above and given in equations (Bngh14) and (Bngh17). Note specifically, that the system resistance can be obtained from steady state operating characteristics; for example, in the case of a pump or turbine, it is directly related to the slope of the head-flow characteristic and instabilities in these devices which result from operation in a regime where the slope of the characteristic is positive and $Re\{-T_{12}\}$ is negative are well known (Greitzer 1981) and have been described earlier (see section on "Surge").

It is, however, important to recognize that the approximate stability criterion $Re\{-T_{12}\} > 0$, while it may provide a useful guideline in many circumstances, is by no means accurate in all cases. One notable and important case in which this criterion is inaccurate is the auto-oscillation phenomenon described in the section on "Auto-oscillation". This is *not* the result of a positive slope in the head-flow characteristic, but rather occurs where this slope is negative and is caused by cavitation-induced changes in the other elements of the transfer function. This circumstance will be discussed further in the section on "Cavitating Inducers".