# Added Mass Matrix

We will consider a finite three-dimensional body moving and accelerating in a fluid at rest far from that body. If that body experiences a general motion with translational accelerations, $A_j$, $j = 1, 2, 3$ in any or all three directions and rotational accelerations, $A_j$, $j = 4, 5, 6$ in the same directions then it will, in general, experience fluid forces, $F_i$, $i = 1, 2, 3$, and moments $F_i$, $i = 4, 5, 6$, in all the same directions. We seek to find the relations between those accelerations and forces. However, unless those relations are linear the construction becomes extremely complicated and not readily addressed analytically. Fortunately there are some fluid flows in which those relations are linear, specifically in the case of potential flow or Stokes's flow at asymptotically small Reynolds numbers. We confine the following discussion to those circumstances. The linear relations allow flows to be superposed and the effects of individual motions and accelerations in one direction to be superimposed on or isolated from those in another direction. We might however note that even in flows which are not superposable, the methodology that follows might be a useful first approximation.

Given superposability and linear relations between the accelerations, $A_j$, $j = 1 \rightarrow 6$, and the forces, $F_i$, $i = 1 \rightarrow 6$, they induce, we can define an **added mass matrix**, $M_{ij}$, as

$$F_i \;=\; -M_{ij}A_j \tag{Bmbb1}$$

When the flow is superposable it is convenient to define $u_{ij}$ as the induced fluid velocities caused by **unit** velocity of the body in the $j$ direction ($j = 1 \rightarrow 6$). Then, if the body velocities are denoted by $U_j$, $j = 1 \rightarrow 6$, it follows that the fluid velocities are

$$u_i \;=\; u_{ij}U_j \tag{Bmbb2}$$

and the total kinetic energy can then be written as

$$T \;=\; \frac{1}{2}A_{jk}U_jU_k \tag{Bmbb3}$$

where the matrix, $A_{jk}$, is composed of elements

$$A_{jk} \;=\; \rho\int_V u_{ij}u_{ik}dV \;=\; M_{jk} \tag{Bmbb4}$$

and it can be shown (Yih 1969, p.102) that the matrix $A_{jk}$ is, in fact, the added mass matrix, $M_{jk}$. It is certainly clear that the diagonal terms, $A_{11}$, $A_{22}$, and $A_{33}$, are identical to the added masses in the introduction (to establish this define the direction $x$ of that introduction as either $x_1$, $x_2$, or $x_3$; then $u_{ij}$ and $u_{ik}$ are identical to the velocity $u_i/U$ in the introduction). Moreover, equation (Bmbb4) also demonstrates that the added mass matrix must be symmetric when the flow is superposable since exchanging the indices $j$ and $k$ in that equation does not change the value of the right hand side. Hence superposability implies symmetry of the added mass matrix. Consequently, in general, the added mass matrix will contain 21 different coefficients, 6 diagonal values and 15 off-diagonal values since a force applied externally to the body will, in general, cause accelerations in all six directions, translational and rotational.

To complete the formulation of the inertial terms in the equation of motion for a body we would add to the left hand side of equation (Bmbb1) the inertial matrix due to the mass and moments of inertia of the material of the body itself. If the center of mass of the body is chosen as the origin for the body mass matrix then that matrix will be symmetric and will contain only 7 different, non-zero values (namely the mass and the six different components of the moment of inertia matrix). This contrasts with the 21 independent coefficients in the added mass matrix. This number can however be reduced when one considers the simplifications caused by geometric symmetries.

The simplifications introduced by geometric symmetries of the body are fairly easily established. Consider, for example, a body with a single plane of symmetry, for example an airplane. It is clearly convenient to select axes such that this plane of symmetry corresponds to the $x_3 = 0$ plane. Then any acceleration confined to this plane, namely any combination of $A_1$, $A_2$ and $A_6$, will produce no added mass force $F_3$, $F_4$ or $F_5$. The only possible non-zero forces will be $F_1$, $F_2$ and $F_6$. It follows that for such a body the following 9 components of the added mass matrix will be zero:

$$M_{ij} \;=\; 0 \quad \text{for} \quad i = 3,4,5 \; ; \; j = 1,2,6 \tag{Bmbb5}$$

If, in addition, the flow is assumed to be potential flow such that the matrix is symmetric then $M_{ji} = 0$ for the same domains of $i$ and $j$. The number of independent, non-zero values required to define the matrix is 12, namely

$$M_{ii} \; , \; i = 1 \rightarrow 6 \quad \text{and} \quad M_{12}, M_{34}, M_{35}, M_{45}, M_{16} \quad \text{and} \quad M_{26} \tag{Bmbb6}$$

Bodies which have two planes of symmetry (for example, a hemisphere) yield a further reduction in the number of non-zero values. Suppose axes are chosen such that both $x_2 = 0$ and $x_3 = 0$ are planes of symmetry. Then not only must equation (Bmbb5) be true but also

$$M_{ij} \;=\; 0 \quad \text{for} \quad i = 2,4,6 \; ; \; j = 1,3,5 \tag{Bmbb7}$$

and again assuming potential flow $M_{ji} = 0$ for the same domains. Then the only non-zero values which need evaluation are

$$M_{ii} \; , \; i = 1 \rightarrow 6 \quad \text{and} \quad M_{35} \quad \text{and} \quad M_{26} \tag{Bmbb8}$$

The last two, which with $M_{62} = M_{26}$ and $M_{53} = M_{35}$ represent the only non-zero off-diagonal terms, correspond to the moment about the $x_3$ axis generated by acceleration in the $x_2$ direction and the moment about the $x_2$ axis generated by acceleration in the $x_3$ direction. In other words since the body is not symmetric about the $x_2x_3$ plane linear acceleration in either the $x_2$ or $x_3$ direction will cause pitching moments in the $x_1x_2$ or $x_1x_3$ planes.

A few simple bodies such as a sphere, circular cylinder, cube, or rectangular box have three planes of symmetry. By following the same procedure used above it is clear that the only possible non-zero elements are

$$M_{ii} \; , \; i = 1 \rightarrow 6 \quad \text{and} \quad M_{15}, M_{16}, M_{24}, M_{26}, M_{34} \quad \text{and} \quad M_{35} \tag{Bmbb9}$$

and that in the case of potential flow all of the off-diagonal terms are zero. Only in this simple case of three axes of symmetry and symmetry of the matrix (see below) does the added mass matrix become purely diagonal so that there are no secondary induced accelerations.

One other form of equation (Bmbb4) is useful in dealing with potential flows. If $\phi_j$ denotes the velocity potential of the steady flow due to motion with unit velocity in the $j$ direction, then it follows that

$$u_{ij} \;=\; \frac{\partial \phi_j}{\partial x_i} \tag{Bmbb10}$$

Then substitution into equation (Bmbb4) and application of Green's theorem yields

$$M_{jk} \;=\; A_{jk} \;=\; -\rho \int_S \phi_j \frac{\partial \phi_k}{\partial n} dS \tag{Bmbb11}$$

where $S$ is the surface of the body and $n$ is the outward normal to that surface. In many potential flows it is clearly easier to evaluate the surface integral in equation (Bmbb11) than the volume integral in equation (Bmbb4).

It is also appropriate to point out that the theoretical values of the added mass for potential flow that are presented in the following sections have a broader relevance than might first be imagined. Even in a viscous fluid, the added mass for a body accelerating from rest in a fluid previously at rest is given by the potential flow value at that first moment when the velocity is still zero. This is because, when the velocity is zero, the vorticity is zero and therefore we can define what is known as the **acceleration potential**, $\phi'$, such that

$$\frac{\partial \underline{u}}{\partial t} = \nabla \phi' \qquad \text{(Bmbb12)}$$

Then conservation of mass for an incompressible fluid leads to Laplace's equation and potential flow for the acceleration. Hence we have potential flow even when the later flow is dominated by viscous effects. It follows that in this first moment the forces and accelerations in a viscous flow are related in the same way as in conventional potential flow.