Turbulence Spectra and Scales

As we described in the section on amplification, transition to turbulence begins when some flow instability (such as the instability analyzed in sections (Bkc) and (Bkd)) leads to some fairly large scale disturbance(s) or "eddies" in the flow field. As these disturbances gather energy from the mean flow, they begin to spawn smaller disturbances or eddies which, in turn spawn even smaller eddies. This process ends because, eventually, the eddies reach a size for which viscous effects become important and the very small eddies are damped out by viscosity. Eventually, the spectrum of spatial or temporal eddy sizes reaches a "fully-developed" state in which energy is fed from the mean flow into large eddies and then continually cascades down to smaller and then smaller eddies eventually reaching a size at which viscosity becomes important and damps out those small eddies. In this fully-developed state the disturbance energy for any one size of eddy becomes relatively constant though it can, of course, continue to change with the flow conditions. To examine this process further, the fluid velocities, stresses and pressures are subdivided into mean, time-averaged quantities denoted by an overbar and unsteady components with zero time averages denoted by a prime:

$$\overline{u}_i \;=\; \overline{u}_i + u_i' \qquad ; \qquad p \;=\; \overline{p}_i + p_i' \tag{Bkf1}$$

and similarly for the individual velocity components, u, v and w, and all the components of the stress tensor, σ_{ij}. To be specific, the mean or overbar steady components are defined by averaging the quantity over a period of time, T, which is much larger than any of the periods of the turbulent fluctuations so that, for example,

$$\overline{u}_i \;=\; \frac{1}{T}\int_t^{t+T} u_i \; dt \tag{Bkf2}$$

so it necessarily follows that

$$\frac{1}{T}\int_t^{t+T} u_i' \; dt \;=\; 0 \tag{Bkf3}$$

The mean kinetic energy (per unit mass) associated with the turbulent motions, u_i', is denoted by E and is given by

$$E \;=\; \frac{1}{2}\overline{u_i' u_i'} \tag{Bkf3}$$

and we visualize this energy as being distributed either spatially over eddies of many sizes (or wavenumbers, k) or over eddies of many frequencies, ω, so that we can define a turbulent energy density, $e(k)$ or $e(\omega)$, such that

$$E \;=\; \int_0^\infty e(k)\; dk \qquad \text{or} \qquad E \;=\; \int_0^\infty e(\omega)\; d\omega \tag{Bkf4}$$

Thus by plotting $e(k)$ against k or $e(\omega)$ against ω we can present a spectrum of the turbulent fluctuations. It was G.I.Taylor who hypothesized that these spatial and temporal distributions were equivalent. A example of a temporal spectrum is presented in Figure 1 in which the amplitude of the turbulent energy density is plotted against the frequency of those fluctuations. The largest eddies (lowest frequencies) on the left side of the spectrum are generated by the mean flow and the energy cascades down to smaller and smaller eddies (larger frequencies) until they are damped out by viscosity.

As another example of a spectrum we include in Figure 2 measurements by Witter *et al.* (2013). Four spectra of the longitudinal velocity are shown for four different heights above the floor of the wind tunnel.

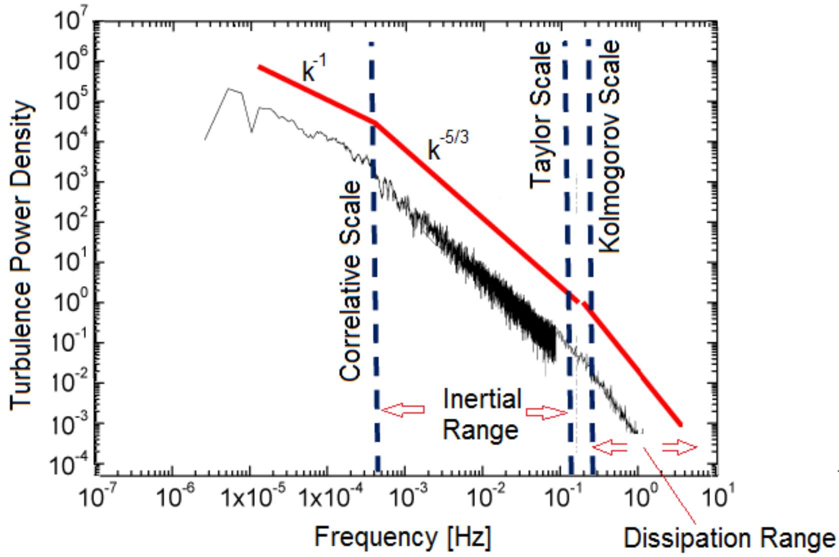

Figure 1: Characteristics of a turbulence spectrum. Adapted from Bruno and Carbone (2013).

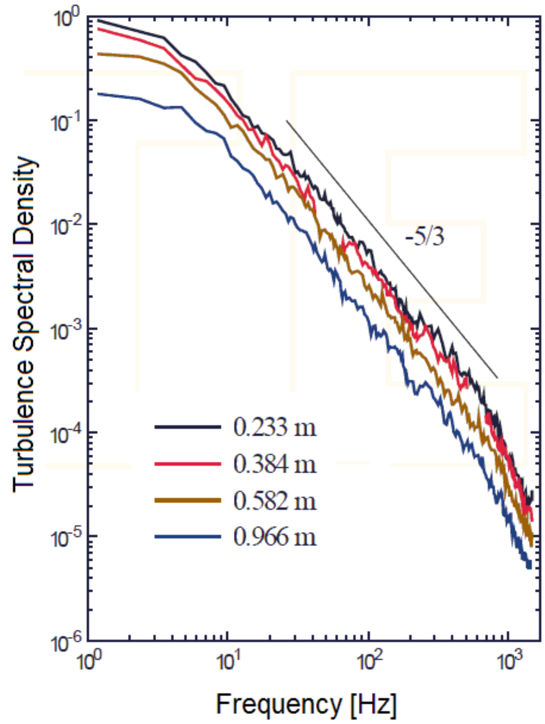

Figure 2: Turbulence spectra measured in a wind tunnel by Witter *et al.* (2013). Four spectra of the longitudinal velocity are shown for four different heights above the floor of the wind tunnel as shown.

Various scales are used to characterize these spectra and processes. G.I.Taylor was the first to use statistical methods to analyse turbulence and to suggest a critical intermediate eddy size below which viscosity would begin to damp out those eddies (called the Taylor microscale). However, it was A.N.Kolmogorov (1941a,b) who substantially advanced the understanding of turbulence by using a combination of physical insight and dimensional analysis to relate the dimensional quantities

- The wavenumber, k, units L^{-1}
- The turbulent kinetic energy per unit mass, E, units L^2/T^2

- The turbulent kinetic energy density per unit mass, $e(k)$, units L^3/T^2
- The turbulent kinetic energy flux per unit mass, ϵ, units L^2/T^3

where the last quantity is defined below.

Kolmogorov argued that there exists an intermediate range of wavenumbers, called the *inertial range*, bounded on the high side by the *injection wavenumber*, k_i, separating the wavenumbers at which the mean flow is creating large eddies from this intermediate, inertial range and bounded on the low side by the *dissipation wavenumber*, k_d, separating the wavenumbers of the inertial range from the wavenumbers that are substantially attenuated by viscosity. He argued that in this inertial range, $k_i < k < k_d$, neither the mean flow creation nor the viscosity are explicitly important, but instead the energy flux down the cascade of eddy size and the local wavenumber, k, are the only controlling parameters. Then, in a steady state, the energy flux, ϵ, flowing down the spectrum to smaller eddies must be relatively constant though it will later decline in the dissipation range, $k > k_d$. Then dimensional analysis requires that, in the inertial range,

$$E \approx C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}} \tag{1}$$

where C is some assumed universal constant. Many turbulent spectra, including those shown in Figures 1 and Tf2, exhibit such an inertial range.

Kolmogorov also argued that the *dissipation wavenumber*, k_d, would be related to the kinematic viscosity, ν, as well the energy flux, ϵ, and therefore by dimensional analysis

$$k_d \propto \nu^{-\frac{3}{4}}\epsilon^{\frac{1}{4}} \tag{2}$$

The inverse of this, λ, is known as the *Kolmogorov length scale* and the corresponding *Kolmogorov time scale* is also given by $(\nu/\epsilon)^{\frac{1}{2}}$ using dimensional analysis.

To provide further perspective on these results, consider a mean flow with velocity, U, and typical dimension, L. The largest eddies would have a velocity and a dimension also given by U and L so that, by dimensional analysis $\epsilon \propto U^3/L$ and hence the *Kolmogorov length scale* would be proportional to $\nu^{\frac{3}{4}}L^{\frac{1}{4}}/U^{\frac{3}{4}}$ so that

$$\frac{\lambda}{L} \propto (Re)^{-\frac{3}{4}} \tag{3}$$

where $Re = UL/\nu$ is the Reynolds number of the flow. This result has important consequences in demonstrating that the higher the Reynolds number of the flow the smaller the significant eddies are relative to the dimension of the flow. Therefore, in order to resolve the details of the flow in a computation with a mesh, the mesh size must become smaller and smaller the larger the Reynolds number.