Reynolds Stresses

In this section we will investigate the effect of the turbulence on the mean flow. It will be assumed that the turbulent spectrum has reached a steady state so that the turbulence can be regarded as *fully developed*. For example, fully developed turbulent pipe flow would have the same statistical properties at any axial location in the pipe.

We wish to identify the mechanism by which the unsteady motions in the turbulence effect the mean motions. To do so the fluid velocities, stresses and pressures are subdivided into mean, time-averaged quantities denoted by an overbar and unsteady components with zero time averages denoted by a prime:

$$u_i \;=\; \overline{u}_i + u_i' \qquad ; \qquad p \;=\; \overline{p}_i + p_i' \tag{Bkg1}$$

and similarly for the individual velocity components, u, v and w, and all the components of the stress tensor, σ_{ij}. To be specific, the mean or overbar steady components are defined by averaging the quantity over a period of time, T, which is much larger than any of the periods of the turbulent fluctuations so that, for example,

$$\overline{u}_i \;=\; \frac{1}{T}\int_t^{t+T} u_i \; dt \tag{Bkg2}$$

so it necessarily follows that

$$\frac{1}{T}\int_t^{t+T} u_i' \; dt \;=\; 0 \tag{Bkg3}$$

We choose to limit the present investigation to incompressible, Newtonian fluids of uniform and constant viscosity for which the equations of continuity and motion (equations (Bcd7), (Bhb7) and (Bhf4)) may be written as

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{Bkg4}$$

and

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} \;=\; \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{Bkg5}$$

where we have omitted the body force term. Substituting the expressions (Bkg1) and then integrating each term over the large time interval, T, using the relations (Bkg2) and (Bkg3) and assuming that the order of the differential and integral operators can be interchanged (an assumption which could introduce some error), we obtain the following expressions that govern the mean motions:

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0 \tag{Bkg6}$$

$$\frac{\partial (\overline{u}_i \overline{u}_j)}{\partial x_j} + \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} \;=\; \frac{1}{\rho}\frac{\partial \overline{\sigma}_{ij}}{\partial x_j} \;=\; -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} \tag{Bkg7}$$

Notice that the integration over T has eliminated all the terms containing a single fluctuation but that the mean of a quadratic combination of fluctuation terms is not necessarily zero and so the convective inertial term containing such a combination of velocities, namely $\partial(\overline{u_i' u_j'})/\partial x_j$, must be retained. Indeed this is the sole contribution of the turbulent fluctuations to the mean motion and will be seen to distinguish a turbulent flow from its laminar counterpart. These additional terms in the governing equations for turbulent flow are called *Reynolds stress terms* for the following reason.

It is noticeable that the additional term in the governing equation (Bkg7) due to the turbulent motions has precisely the same mathematical form as the stress terms. It is therefore convenient to combine these and write equation (Bkg7) as

$$\overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial (\overline{\sigma}_{ij} - \rho \overline{u_i' u_j'})}{\partial x_j} \tag{Bkg8}$$

where we have also used the continuity equation (Bkh6). Consequently the effect of the turbulent motions is to add additional "stresses" to what are otherwise the same equations that govern laminar flow. These additional "stresses" are called Reynolds stresses though it is important to recognize that they are not stresses but are instead additional momentum fluxes due to the unsteady turbulent motions. Thus the effective total stress tensor in the turbulent flow denoted by σ_{ij}^* is given by

$$\sigma_{ij}^* = \overline{\sigma}_{ij} - \rho \overline{u_i' u_j'} \tag{Bkg9}$$

and written out in its components this represents the following individual stresses:

$$\sigma_{xx}^* = \overline{\sigma}_{xx} - \rho \overline{u'^2} \quad ; \quad \sigma_{yy}^* = \overline{\sigma}_{yy} - \rho \overline{v'^2} \quad ; \quad \sigma_{zz}^* = \overline{\sigma}_{zz} - \rho \overline{w'^2} \tag{Bkg10}$$

where the second terms are called the *Reynolds normal stresses* and

$$\sigma_{xy}^* = \overline{\sigma}_{xy} - \rho \overline{u'v'} \quad ; \quad \sigma_{yz}^* = \overline{\sigma}_{yz} - \rho \overline{v'w'} \quad ; \quad \sigma_{zx}^* = \overline{\sigma}_{zx} - \rho \overline{w'u'} \tag{Bkg11}$$

where the second terms are called the *Reynolds shear stresses.*

It is instructive to demonstrate how the turbulent motions give rise to additional momentum fluxes and can then be interpreted as additional effective stresses. As an example we choose to focus on just one of the Reynolds stresses, in particular the shear stress, $\rho \overline{u'v'}$. Consider the elemental control volume, $dxdydz$, sketched in Figure 1 and placed in a turbulent flow such that $\overline{v} = \overline{w} = 0$ and $\overline{u} \neq 0$. We anticipate that

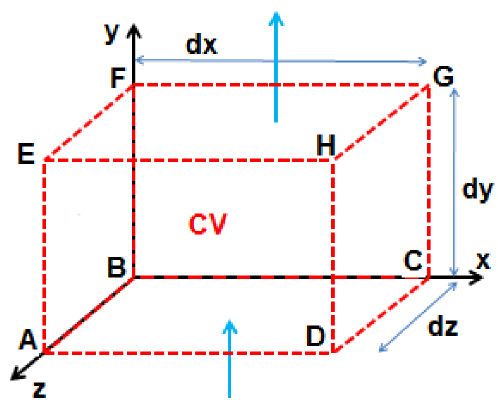

Figure 1: An elemental control volume placed in a turbulent flow such that $\overline{v} = \overline{w} = 0$ and $\overline{u} \neq 0$.

$\rho \overline{u'v'}$ is associated with the fluxes of x-momentum (momentum in the x direction) through the sides ABCD and EFGH. The instantaneous flux of x-momentum into the control volume through ABCD is

$$\rho dxdzv' \times (\overline{u} + u') \tag{Bkg12}$$

and the instantaneous flux of x-momentum out of the control volume through EFGH is

$$\rho dxdz \left\{ v'(\overline{u} + u') + dy \frac{\partial}{\partial y} \left(v'(\overline{u} + u') \right) \right\} \tag{Bkg13}$$

and therefore the net flux of x-momentum out of the control volume through the sides ABCD and EFGH is

$$\rho dxdydz\frac{\partial}{\partial y}\left(v'(\overline{u}+u')\right) \tag{Bkg14}$$

and the mean or time-averaged component of this net flux of x-momentum out of the control volume is

$$\rho dxdydz\frac{\partial}{\partial y}\left(\overline{u'v'}\right) \tag{Bkg15}$$

Now an additional shear stress, $\Delta\sigma_{xy}$, acting on this control volume would produce a force $-dxdz\Delta\sigma_{xy}$ acting on the surface ABCD in the x direction and a force, $dxdz\left\{\Delta\sigma_{xy}+dy\partial/\partial y(\Delta\sigma_{xy})\right\}$ acting on the surface EFGH and so yield a net force in the x direction equal to

$$dxdydz\frac{\partial\Delta\sigma_{xy}}{\partial y} \tag{Bkg16}$$

Therefore, the net flux of x-momentum given by the expression (Bkg15) is equivalent to the negative of the expression (Bkg16) and it follows that

$$\Delta\sigma_{xy} \;=\; -\rho\overline{u'v'} \tag{Bkg17}$$

and is the Reynolds shear stress. Similar explanations follow for all the other Reynolds stresses.

To summarize, the equations for the steady component of fully-developed turbulent flows are

$$\frac{\partial\overline{u}_j}{\partial x_j} \;=\; 0 \tag{Bkg18}$$

$$\overline{u}_j\frac{\partial\overline{u}_i}{\partial x_j} \;=\; -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x_i}+\nu\frac{\partial^2\overline{u}_i}{\partial x_j\partial x_j}-\frac{\partial}{\partial x_j}(\overline{u_i'u_j'}) \tag{Bkg19}$$

and written out in components for the planar flows that will be the focus of the following sections these become

$$\frac{\partial\overline{u}}{\partial x}+\frac{\partial\overline{v}}{\partial y} \;=\; 0 \tag{Bkg20}$$

$$\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y} \;=\; -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial x}+\nu\left\{\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}\right\}-\frac{\partial}{\partial x}(\overline{u'^2})-\frac{\partial}{\partial y}(\overline{u'v'}) \tag{Bkg21}$$

$$\overline{u}\frac{\partial\overline{v}}{\partial x}+\overline{v}\frac{\partial\overline{v}}{\partial y} \;=\; -\frac{1}{\rho}\frac{\partial\overline{p}}{\partial y}+\nu\left\{\frac{\partial^2\overline{v}}{\partial x^2}+\frac{\partial^2\overline{v}}{\partial y^2}\right\}-\frac{\partial}{\partial y}(\overline{v'^2})-\frac{\partial}{\partial x}(\overline{u'v'}) \tag{Bkg22}$$

In order to proceed to solve these equations for a turbulent flow we must find some way to relate the Reynolds stresses to the mean flow velocities. Such relations are known as turbulence models and these will be addressed in section (Bkh).

First, however, it is appropriate to consider the magnitudes of the Reynolds stress terms, whether all of them are significant and where they might be most substantial. To do this we should compare them with other, mean flow terms in the equations of motion. Consider, for example, the two Reynolds stress terms in equation (Bkg21); the penultimate term is a Reynolds normal stress term and the last is a Reynolds shear stress term. The appropriate comparison for the normal stress term is with the first two terms in the equation, the primary inertial terms. It is clear that the typical magnitude of the Reynolds normal stress term to these primary inertial terms is u'^2/U^2 where U is the typical mean velocity in the flow. Typical relative magnitudes are shown in the data from a turbulent boundary layer presented in Figure 2. Since u'^2/U^2, v'^2/U^2 and w'^2/U^2 are all very small (of order 0.004), it is clear at least in the case of a turbulent

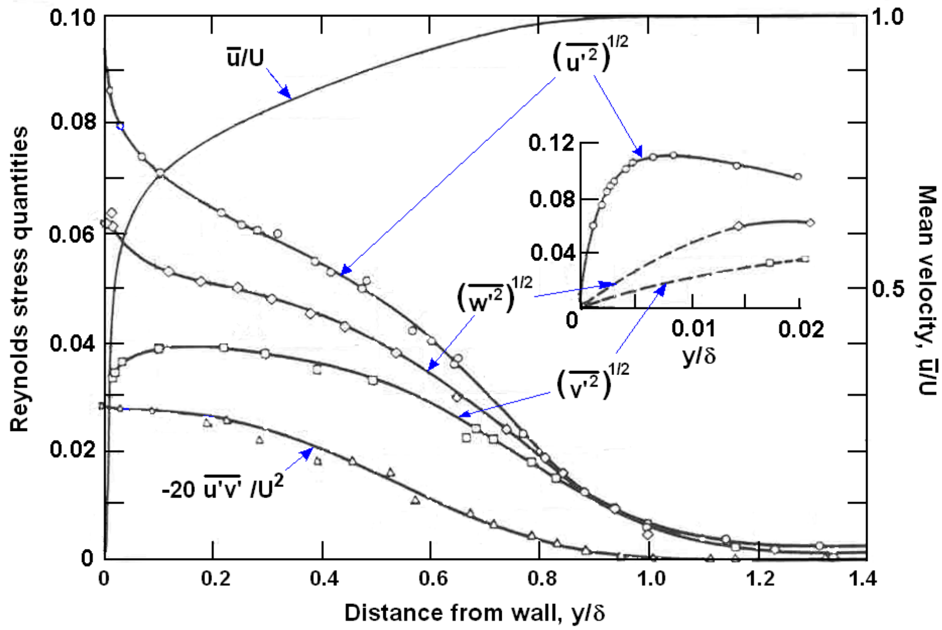

Figure 2: Measurements of the Reynolds stresses in a turbulent boundary layer.

boundary layer that the Reynolds normal stress terms are very small compared with the primary inertial terms. Consequently, the Reynolds normal stress terms are usually neglected in turbulence models.

In contrast, the appropriate comparison for the Reynolds shear stress term is with the other viscous shear stress terms and here the relative magnitude will be given by

$$\frac{\overline{u'v'}}{\nu \frac{\partial \overline{u}}{\partial y}} \approx \frac{\overline{u'v'}}{U^2}\frac{U\delta}{\nu} \tag{Bkg23}$$

As can be seen in Figure 2, $\overline{u'v'}/U^2$ is small (order 0.001) but $\frac{U\delta}{\nu}$ could be large and so, at least in case of a turbulent boundary layer, the Reynolds shear stress terms cannot be neglected.