Linear Stability Analyses

The stability of steady laminar flows of an incompressible, Newtonian fluid of kinematic viscosity, ν, is assessed by a linear stability analysis that begins with the relevant equations of continuity and motion, (Bhf3) and (Bhf4):

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(Bkc1)}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad \text{(Bkc2)}$$

Each of the flow variables, q (where q represents u_i or p) is then decomposed into a steady or time-independent component denoted by an overbar, $\bar{q}$, and a small, time-dependent perturbation, $\tilde{q}$. The latter is envisaged as a small perturbation such that $\tilde{q} \ll \bar{q}$ so that terms that are quadratic (or higher order) in $\tilde{q}$ quantities can be neglected leaving only terms that are either independent of or linear in $\tilde{q}$ quantities. Consequently various perturbations, $\tilde{q}$, can be linearly superposed.

We will conduct what is known as a *normal mode analysis* by considering perturbations that are oscillatory in time, t, and in one spatial direction, say x, so that $\tilde{q}$ may be written in the form

$$q = \bar{q}(x_i) + \tilde{q}(x_i, t) = \bar{q}(x_i) + Re\left\{\tilde{q}^*(x_i)e^{i(kx-\omega t)}\right\} \qquad \text{(Bkc3)}$$

where i is the square root of -1, $Re\{\ \ \}$ denotes the "real part of" and $\tilde{q}^*$ is the amplitude of the perturbation. The wavenumber, k, and the radian frequency, ω, may both be complex so that

$$k = k_R + ik_I \qquad \text{and} \qquad \omega = \omega_R + i\omega_I \qquad \text{(Bkc4)}$$

where the subscripts R and I denote the real and imaginery parts. This allows two types of solution that are of particular interest namely

- oscillatory perturbations that have an amplitude that is growing in space but not in time so that $\omega_I = 0$ and $\omega = \omega_R$ is the perturbation frequency. Then k_R is the wavenumber of the wave-like perturbation and k_I is the growth or attenuation rate. For convenience we refer to this as the *spatial growth* case.
- oscillatory perturbations that have an amplitude that is growing in time but not in space so that $k_I = 0$ and $2\pi/k = 2\pi/k_R$ is the wavelength of the perturbation. Then ω_R is the spatial wavenumber of the wave-like perturbation and ω_I is the spatial growth or attenuation rate. For convenience we refer to this as the *temporal growth* case.

Here we will focus primarily on the first case and examine the rate of growth, ω_I, of perturbations of frequency, ω_R, and wavenumber, k_R.

When expansions of the form (Bkc3) are substituted into the governing equation (Bkc2), the terms which are independent of t are isolated and solved to obtain the mean motion. The terms that are linear in the perturbations $\tilde{q}^*$ are the linear stability equations that are to be solved to determine the stability of the flow.

In practice, the implementation of this procedure is very difficult unless the basic mean flow is simple. Here we shall limit the implementation to simple planar, parallel flows in which the unperturbed flow consists of a velocity, $\bar{u} = U(y)$, in the x direction, $\bar{v} = 0$ and $\bar{p}$ is uniform and constant. The equations of motion (Bkc2) for this parallel flow are:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} \tag{Bkc5}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right\} \tag{Bkc6}$$

The perturbations in the streamfunction, $\tilde{\psi}$, and pressure, $\tilde{p}$, will be represented by

$$\tilde{\psi} = Re\left\{f(y)e^{i(kx-\omega t)}\right\} \qquad \text{and} \qquad \tilde{p} = Re\left\{g(y)e^{i(kx-\omega t)}\right\} \tag{Bkc7}$$

so that

$$\tilde{u} = Re\left\{\frac{df}{dy}e^{i(kx-\omega t)}\right\} \qquad \text{and} \qquad \tilde{v} = Re\left\{-ikf(y)e^{i(kx-\omega t)}\right\} \tag{Bkc8}$$

and substituting these expressions into the equations of motion, (Bkc5) and (Bkc6), yields

$$(ikU - i\omega)\frac{df}{dy} - ik\frac{dU}{dy}f = -\frac{ikg}{\rho} + \nu\left\{-k^2\frac{df}{dy} + \frac{d^3 f}{dy^3}\right\} \tag{Bkc9}$$

$$(k^2U - k\omega)f = -\frac{1}{\rho}\frac{dg}{dy} + \nu\left\{ik^3 f - ik\frac{d^2 f}{dy^2}\right\} \tag{Bkc10}$$

Eliminating the function g from these two equations results in

$$(\omega - kU)\left\{\frac{d^2 f}{dy^2} - k^2 f\right\} + kf\frac{d^2U}{dy^2} = i\nu\left\{\frac{d^4 f}{df^4} - 2k^2\frac{d^2 f}{dy^2} + k^4 f\right\} \tag{Bkc11}$$

This equation which must be solved for the perturbation $f(y)$ is called the *Orr-Sommerfeld equation*. The version in which the viscous terms are neglected is

$$(\omega - kU)\left\{\frac{d^2 f}{dy^2} - k^2 f\right\} + kf\frac{d^2U}{dy^2} = 0 \tag{Bkc12}$$

and is called the *Rayleigh equation*. Notice that both versions are homogeneous in f and represent eigenvalue problems for which we need to identify boundary conditions.

It remains to discuss the boundary conditions under which these equations must be solved. We will do so for boundary layer velocity profiles, for planar Couette flow and for planar pipe flow, cases which will be addressed in other sections. At any solid boundary parallel with the x direction at, say, $y = 0$, the zero normal velocity and no-slip conditions require that

$$(f)_{y=0} = 0 \qquad \text{and} \qquad \left(\frac{df}{dy}\right)_{y=0} = 0 \tag{Bkc13}$$

Furthermore, in the case of the boundary layer problem we must require that

$$(f)_{y\to\infty} \rightarrow 0 \tag{Bkc14}$$

in order that the perturbation velocities decay to zero at large y. In the inviscid case governed by the Rayleigh equation (Bkc12), these three boundary conditions (equations (Bkc13) and (Bkc14)) complete the eigenvalue problem. The calculation for the spatial problem using a shooting method can proceed as follows. Given $U(y)$ and a real frequency, ω_R, we choose guessed vales for both k_R and k_I and begin the numerical integration for f at $y = 0$ where both f and df/dy are zero. The Rayleigh equation then yields d^2f/dy^2 and we integrate using, for example, a Runge-Kutta procedure to find f and df/dy at the next y mesh point (this is best done along a complex y contour in order to avoid potential singularities on the real y axis). As the integration approaches large and real values of y the boundary condition as $y \rightarrow \infty$ must be satisfied and this provides a criterion by which both k_R and k_I must be adjusted in order to satisfy that condition. An iterative method can then be used to determine the final values of k_R and k_I. Note that the temporal problem for a particular k_R can be addressed in a precisely analogous way except that this involves initial guessed and finally determined values of ω_R and ω_I.

Planar Couette flow or planar pipe flow can be handled in a manner very similar to the above procedures for the boundary layer flow except that the boundary condition at $y \rightarrow \infty$ is now replaced by the condition that $f = 0$ at $y = h/2$ where h is the width of the gap or pipe.

The Orr-Sommerfeld equation contains fourth order terms in addition to the second order terms in the inviscid Rayleigh equation and therefore the viscous calculation using the Orr-Sommerfeld equation requires two additional boundary conditions, specifically values of d^2f/dy^2 and d^3f/dy^3 at $y = 0$. Usually it is assumed that these derivatives at $y = 0$ are zero.

In the next section, typical results from these instability calculations are presented and discussed.