Incompressible, Inviscid, Irrotational Flow

As described earlier, irrotational flow is defined as a flow in which the vorticity, $\underline{\omega}$, is zero and since

$$\underline{\omega} = \nabla \times \underline{u} \quad \text{(Bga1)}$$

it follows that the condition, $\underline{\omega} = 0$, is automatically satisfied by defining a quantity called the **velocity potential**, ϕ, such that

$$\underline{u} = \nabla \phi \quad \text{(Bga2)}$$

since it is always true that $\nabla \times \nabla \phi = 0$. For this reason irrotational flow is often called **potential flow** and we will refer to it as such. Other forms of equation (Bga2) are

$$u_i = \frac{\partial \phi}{\partial x_i} \quad ; \quad u = \frac{\partial \phi}{\partial x} \, , \, v = \frac{\partial \phi}{\partial y} \, , \, w = \frac{\partial \phi}{\partial z} \quad \text{(Bga3)}$$

In addition, if the fluid is incompressible, then conservation of mass requires that

$$\nabla . \underline{u} \; = \; 0 \quad \text{or} \quad \frac{\partial u_j}{\partial x_j} \; = \; 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \; = \; 0 \quad \text{(Bga4)}$$

and substituting equation (Bga2) into (Bga4) yields

$$\nabla^2 u = 0 \quad \text{(Bga5)}$$

These are the governing equations of incompressible, inviscid, potential flow and we will explore the characteristics of some of these flows in the following pages; in particular we will examine many planar flows.

In planar flow the relations (Bga3) allow us to define the streamfunction, ψ, such that in incompressible planar flow in the xy plane

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \quad \text{(Bga6)}$$

and these relations taken together with

$$u = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} \quad \text{(Bga7)}$$

are called the Cauchy-Riemann equations. They imply that, in incompressible planar potential flow, the lines of constant velocity potential or equipotentials are everywhere perpendicular to the lines of constant ψ, namely streamlines. Also note that substituting the relations (Bga5) into $\nabla \times \underline{u} = 0$ yields

$$\nabla^2 \psi \; = \; 0 \quad \text{or} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \text{(Bga8)}$$

These are the governing equations of planar, incompressible, inviscid, potential flow and we will explore the characteristics of many of these flows in the sections which follow.

It follows from the preceding, that the solution of incompressible, inviscid, potential flow requires the solution of Laplace's equation

$$\nabla^2 \phi \; = \; 0 \quad \text{(Bga9)}$$

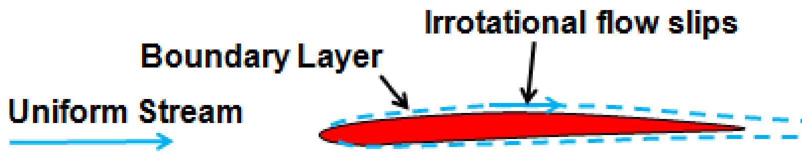

Figure 1: Thin boundary layer at large Reynolds numbers.

which would yield ϕ and therefore the velocity components from equation (Bga3). Once the velocity components and the velocity magnitude have been obtained, the pressure p would follow from Bernoulli's equation which is applicable to all these potential flows. Of course, the solution of equation (Bga9) also requires boundary conditions and these will depend on the nature of the boundary. At a solid boundary, as discussed in the section on vorticity, there is a thin boundary layer (at least in high Reynolds number flows) within which the flow is **not** irrotational. Consequently we visualize solving for the irrotational flow outside of this layer. However, if the layer is very thin (and this applies under conditions discussed at length in the sections on boundary layers), we can assume that the surface on the outer edge of the boundary layer is approximately the same shape as that of the surface (see Figure 1) and therefore that the condition of zero velocity normal to the surface also applies to the boundary of the potential flow. Under these conditions the boundary condition for potential flow at a solid boundary is

$$\underline{u}.\underline{n} = 0 \text{ or } \frac{\partial \phi}{\partial n} = 0 \qquad \text{(Bga10)}$$

where the vector, $\underline{n}$, and coordinate, n, are normal to the solid surface. On the other hand, the usual no slip condition of zero tangential velocity which applies at the actual solid surface cannot and should not be applied at the boundary of the potential flow since the velocity at the outer edge of the boundary layer will not be zero.

There are a number of methods available for the task of solving Laplace's equation, the first step in generating a solution to a potential flow, and we will discuss these in some detail as we progress through a series of examples of solutions to incompressible, inviscid, irrotational flows.