Axisymmetric Flow without Swirl

As described in the preceding section (Bgfa), in the absence of swirl ($u_\theta = 0$), the equations for axisymmetric potential flow are

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{Bgfb1}$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{Bgfb2}$$

and we can define Stokes' stream function, ψ, such that

$$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \quad ; \quad u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{Bgfb3}$$

which ensures that continuity is satisfied.

Eliminating of the pressure from equations (Bgfb1) and (Bgfb2) leads, as expected, to the appropriate form of the vorticity transport equation for this type of flow, namely,

$$\frac{D(r\omega_\theta)}{Dt} = 0 \tag{Bgfb4}$$

where, as defined in equation (Bgfa4), the only non-zero vorticity component in this type of flow, ω_θ, is given by

$$r\omega_\theta = r\left\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right\} = -\frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{Bgfb5}$$

If the flow is steady it follows that solution of a steady, axisymmetric flow without swirl requires the solution of Laplaces's equation

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0 \tag{Bgfb6}$$

This is much more difficult to solve than the equivalent Laplace's equation for steady, planar, potential flow and the methods for doing so are much more limited that the methods for the planar flow in sections (Bga) through (Bgd). One method that was used for axisymmetric flows in the years before numerical methods, particularly for the design of axisymmetric airships, is to distribute along the axis various point singularities of the types described and defined in section (Bgdn). In that respect and in the airship application, a distribution of point doublets is to be preferred over a distribution of sources and sinks since the doublet distribution ensures a finite body.