Axisymmetric Flow

We now turn to inviscid, incompressible, axisymmetric potential flow. Using cylindrical coordinates, (r, θ, z), where $r = 0$ is the axis of the axisymmetric flow and (u_r, u_θ, u_z) are the velocities in those (r, θ, z) directions the continuity equation (see equation (Bce11)) is

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_z)}{\partial z} \;=\; 0 \tag{Bgfa1}$$

and this allows the definition of another stream function, ψ, known as Stokes' stream function (different from the stream function used in planar flow) defined as

$$u_z \;=\; \frac{1}{r}\frac{\partial \psi}{\partial r} \quad ; \quad u_r \;=\; -\frac{1}{r}\frac{\partial \psi}{\partial z} \tag{Bgfa2}$$

and whose definition automatically assures that the continuity equation (Bgfa1) is satisfied.

For future reference we also note that the vorticity components in incompressible axisymmetric flow (see equations (Bba27) to (Bba29) with $\partial/\partial\theta$ terms set to zero) are

$$\omega_r \;=\; -\frac{\partial u_\theta}{\partial z} \tag{Bgfa3}$$

$$\omega_\theta \;=\; \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{Bgfa4}$$

$$\omega_z \;=\; \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} \tag{Bgfa5}$$

and in the absence of swirl ($u_\theta = 0$) only the θ component remains:

$$\omega_r = \omega_z = 0 \quad \text{and} \quad \omega_\theta \;=\; \omega \;=\; \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \tag{Bgfa6}$$

Deleting the viscous terms and absorbing the force field terms into the pressure, the equations of motion (equations (Bhg1) to (Bhg3)) yield

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{Bgfa7}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} \;=\; -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \tag{Bgfa8}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{Bgfa9}$$

Note that the last terms on the left hand sides of equations (Bfga7) and (Bfga8), namely u_θ^2/r and $u_\theta u_r/r$, are due to the centripetal and Corioli's components of acceleration. Setting $\partial/\partial\theta$ terms equal to zero for axisymmetric flow, equations (Bgfa7) to (Bgfa9) become

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \;=\; -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{Bgfa10}$$

$$\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} = 0 \qquad \text{(Bgfa11)}$$

$$\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{(Bgfa12)}$$

Moreover, if the flow is *steady*:

$$u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{(Bgfa13)}$$

$$u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + \frac{u_\theta u_r}{r} = 0 \qquad \text{(Bgfa14)}$$

$$u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{(Bgfa15)}$$

At this point we note that, in cylindrical coordinates, the Lagrangian derivative (equation (Bab2)) is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z} \qquad \text{(Bgfa16)}$$

which, for axisymmetric flow, becomes

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + u_z\frac{\partial}{\partial z} \qquad \text{(Bgfa17)}$$

and therefore the equation of motion in the θ direction, equation (Bgfa11) or (Bgfa14), can be written as

$$\frac{Du_\theta}{Dt} + \frac{u_\theta u_r}{r} = 0 \qquad \text{(Bgfa18)}$$

which simply states the following: if there is no swirl at any point on a given streamline so that $u_\theta = 0$, then the swirl will be zero all along that streamline. Moreover if there is swirl on a given streamline so that $u_\theta \neq 0$ then the value of u_θ on that streamline will only change due to stretching of the vorticity as the radius increases (or the reverse). We note that both of these results also follow from Kelvin's theorem described and derived in section (Bdj). To see this, consider the circulation Γ around a contour that is a circle of radius r whose center lies on the axis of the axisymmetric flow and whose plane is perpendicular to that axis. Therefore $\Gamma = 2\pi r u_\theta$ and from Kelvin's theorem

$$\frac{D\Gamma}{Dt} = 0 \qquad \text{(Bgfa19)}$$

it follows that

$$\frac{D(ru_\theta)}{Dt} = 0 \qquad \text{(Bgfa20)}$$

which, after manipulation using the expression for the Lagrangian derivative, leads directly to the result (Bgfa18).

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The sections which follow detail some of the characteristics of both axisymmetric flows without swirl and axisymmetric flows with swirl.