Method of Complex Variables for Planar Potential Flows

One of the most powerful tools for the solution of planar potential flows is the method of complex variables. This is based on the so-called Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{Bgea1}$$

which have the following mathematical consequence. If we define a complex position vector,

$$z = x + iy = re^{i\theta} \tag{Bgea2}$$

and a complex potential $f = \phi + i\psi$ then it follows from the Cauchy-Riemann equations that any function $f(z)$ is necessarily a solution of Laplace's equation

$$\nabla^2 \phi = 0 \quad \text{and} \quad \nabla^2 \psi = 0 \tag{Bgea3}$$

To prove this we replace the independent variables x and y by the variable $z = x + iy$ and its complex conjugate $\overline{z} = x - iy$ so that in general $f(z, \overline{z})$ will be a function of both z and $\overline{z}$. Moreover since

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i} \tag{Bgea4}$$

then

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial \overline{z}} = \frac{1}{2} \quad \text{and} \quad \frac{\partial y}{\partial z} = -\frac{\partial y}{\partial \overline{z}} = \frac{1}{2i} \tag{Bgea5}$$

If we then examine the derivative:

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \left\{\frac{\partial f}{\partial x}\right\}\left\{\frac{1}{2}\right\} + \left\{\frac{\partial f}{\partial y}\right\}\left\{-\frac{1}{2i}\right\} \tag{Bgea6}$$

and therefore

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right\} + \frac{i}{2}\left\{\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right\} \tag{Bgea7}$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left\{\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}\right\} + \frac{i}{2}\left\{\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right\} = 0 \tag{Bgea8}$$

because of the Cauchy-Riemann relations. Since $\partial f/\partial \overline{z} = 0$ it follows that f is only a function of z and not of $\overline{z}$.

It therefore follows that any function $f(z)$ that satifies the Cauchy-Riemann relations, therefore satisfies $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$ and therefore constitutes the solution to a planar potential flow. It may, of course, take some investigation to determine what potential flow corresponds to a particular $f(z)$. But the inverse problem of determining the $f(z)$ for a particular potential flow is more difficult so we focus first on the direct problem of determining the flow that corresponds to a chosen, $f(z)$. To aid in that process we note that the derivative

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \left\{\frac{\partial f}{\partial x}\right\}\left\{\frac{1}{2}\right\} + \left\{\frac{\partial f}{\partial y}\right\}\left\{\frac{1}{2i}\right\} \tag{Bgea9}$$

and therefore

$$\frac{df}{dz} = \frac{1}{2}\left\{\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\right\} + \frac{i}{2}\left\{-\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}\right\} = u - iv \tag{Bgea10}$$

so that simple differentiation of the function $f(z)$ yields the velocity components, u and v. In the sections which follow we explore the flows that correspond to particular choices of the function $f(z)$.