Newton's Law of Motion

Having established the form of the first basic conservation law (namely the conservation of mass) in the context of fluid flow we now turn to the second basic conservation principle, namely Newton's first law of motion, and explore the form it takes when applied to a flowing fluid. Newton's law states that the net vector force, $\underline{F}$, on a specific mass (of fluid, solid or any combination thereof) is equal to the rate of change of momentum of that mass. It is particular important to note that this applies to a Lagrangian mass, M, a particular group of particles whose motion is being followed in the flow. We write this as

$$\underline{F} = \frac{D\{M\underline{u}\}}{Dt} = M\frac{D\underline{u}}{Dt} \quad \text{(Bda1)}$$

where $\underline{u}$ is the vector fluid velocity and the time derivative, D/Dt, is the Lagrangian derivative following the fluid (or fluid and solid). The second form on the right of the above equation follows since the mass, M, in a Lagrangian volume does not change with time.

As in the case of the development of the equation for conservation of mass, we will develop several applications of Newton's law using both infinitesmal and macroscopic control volumes. We begin utilizing an infinitesmal control volume to develop differential equations that embody Newton's law and we do this at two levels of complexity, one which neglects viscous forces and leads to Euler's equations and the second which includes those viscous forces and leads to the Navier-Stokes equations. Later we utilize a macroscopic control volume to develop the very useful momentum thereoms of fluid mechanics, tools that are particularly useful to the engineer.

We begin by applying Newton's law to the infinitesmal control volume $dx \times dy \times dz$ shown in Figure 1 which contains a mass of fluid $\rho\ dxdydz$ so that Newton's law can be written as

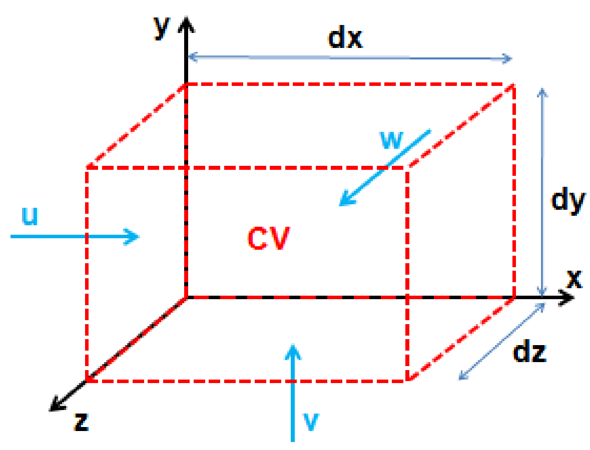

Figure 1: Infinitesmal Eulerian control volume.

$$\underline{F} = \rho dxdydz\frac{D\underline{u}}{Dt} \quad \text{(Bda2)}$$

It is useful to delineate several forms of this equation. In tensor form it may be written as

$$\frac{F_i}{dxdydz} = \rho\frac{Du_i}{Dt} = \rho\left\{\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right\} \quad \text{(Bda3)}$$

where we have used the relation between the Lagrangian and Eulerian time derivatives to write the second version. Note for future reference that $F_i/dxdydz$ is the net force (per unit volume) acting on the infinitesmal control volume and it remains to develop that quantity.

It is also useful to further develop the vector form of the above equations, namely

$$\frac{\underline{F}}{dxdydz} = \rho \left\{ \frac{\partial \underline{u}}{\partial t} + (\underline{u}\ .\nabla)\underline{u} \right\} \tag{Bda4}$$

by utilizing the following vector identity:

$$\frac{1}{2}\nabla(\underline{u}\ .\ \underline{u}) \ = \ \nabla\left(\frac{|\underline{u}|^2}{2}\right) \ = \ \underline{u} \times (\nabla \times \underline{u}) + (\underline{u}\ .\ \nabla)\underline{u} \tag{Bda5}$$

so that the vector form of Newton's law can also be written as

$$\frac{\underline{F}}{dxdydz} = \rho \left\{ \frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{|\underline{u}|^2}{2}\right) \ - \ \underline{u} \times (\nabla \times \underline{u}) \right\} \tag{Bda6}$$

It remains to evaluate the net force $\underline{F}$ acting on the control volume. This consists of a number of contributions divided into two categories, **body forces** and **surface forces**. The so-called **body forces** such as gravity or electromagnetic forces act on the body of fluid inside the control volume. Electromagnetic forces are not presently included in this text and we shall include only gravity in the present development. If the body force per unit volume has components, f_i, then that contribution to $F_i/dxdydz$ is simply f_i. If, like gravity, the body force is **conservative** (the energy expended in moving a mass from one location to another is all recovered when the mass is returned to its original location) then we can define a **body force potential** such that $\underline{f} = \nabla(\mathcal{U})$ and $f_i = \partial\mathcal{U}/\partial x_i$. If, for example, we define a set of axes such that y is vertically upward (this is a common choice and universally the case in this text) then it follows that

$$\mathcal{U} \ = \ -\ \rho g y \quad , \quad f_y \ = \ -\rho g \quad , \quad f_x \ = \ f_z \ = \ 0 \tag{Bda7}$$

where g is the acceleration due to gravity. In the sections which follow the only body force that we will include will be that due to gravity and the above expressions for $\mathcal{U}$ and f_i will be deployed.

There are also **surface forces** which act on the surfaces of the control volume. Principle among theses are the forces which the surrounding fluid is imposing on the fluid inside the control volume. Depending on the nature of the fluid these surface forces can be quite complex. It is convenient to begin by developing the equations in circumstances in which the surfaces forces are assumed to be simple and specifically consist only of forces due to the pressure imposed by the surrounding fluid on the faces of the control volume. Those forces are normal to the faces on which they act. The result will be the set of equations of motion known as **Euler's equations**. Those equations omit the tangential forces that act on the surfaces of the control volume which are usually shear stresses caused by the viscosity of the fluid. Later in this text we return to the control volume analysis to include these viscous forces; the resulting set of equations of motion are called the **Navier-Stokes equations**.