Euler's Equations of Motion

As previously derived, Newton's first law of motion applied to the infinitesmal control volume $dx \times dy \times dz$

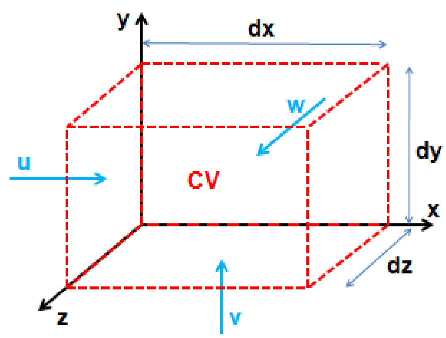

Figure 1: Infinitesmal Eulerian control volume.

shown in Figure 1 can be written in tensor form as

$$\frac{F_i}{dxdydz} = \rho \left\{ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right\} \qquad \text{(Bdb1)}$$

or equivalently in vector form as

$$\frac{\underline{F}}{dxdydz} = \rho \left\{ \frac{\partial \underline{u}}{\partial t} + (\underline{u} \,.\nabla)\underline{u} \right\} \qquad \text{(Bdb2)}$$

where $F_i/dxdydz$ is the force acting on the control volume divided by its volume. In the absence of tangential surface forces, $F_i/dxdydz$ will consist of body forces f_i plus surface forces imposed by the surrounding fluid. Euler's equations are that version of the equations of motion which neglect any tangential surface forces and include only the normal forces, the forces due to the pressure, p. In a Newtonian fluid this implies that we are assuming an inviscid fluid in which all viscous forces are neglected and, since tangential forces are proportional to the viscosity, the tangential forces are zero.

Under these circumstances we need only assess the net pressure forces acting on the control volume in

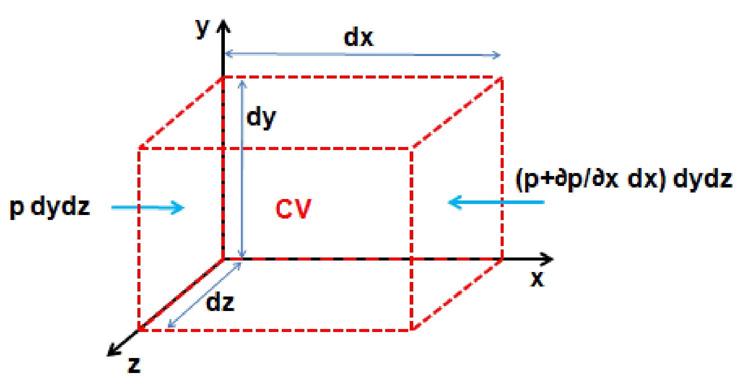

Figure 2: The forces due to pressure on faces normal to the x direction.

order to complete the derivation of Euler's equations. It is simplest to do this one Cartesian component at a time and we choose first the x direction. Since the only forces due to pressure acting in the x direction are those on the faces of the control volume perpendicular to the x direction we need only consider the forces indicated in Figure 2. If we define the pressure acting on the left hand side of the control volume as p then the force acting on that side is $p\ dydz$ in the positive x direction. It follows that the pressure acting on the right hand side is $[p + (\partial p/\partial x)dx]$ and therefore the force acting on the right hand side is $[p + (\partial p/\partial x)dx]dydz$ in the negative x direction. Consequently the net force on the control volume in the positive x direction is $[-(\partial p/\partial x)dxdydz]$ and the force per unit volume is $-(\partial p/\partial x)$ and this is the contribution to $F_x/dxdydz$ due to the pressure acting on the control volume. Similarly the contributions to $F_y/dxdydz$ and $F_z/dxdydz$ are $-(\partial p/\partial y)$ and $-(\partial p/\partial z)$ respectively. Finally Euler's equations of motion for an inviscid fluid become

$$\rho \left\{ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right\} = -\frac{\partial p}{\partial x_i} + f_i \tag{Bdb3}$$

or equivalently in vector form

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + (\underline{u}\ .\nabla)\underline{u} \right\} = -\nabla p + \underline{f} \tag{Bdb4}$$

Another useful vector form results from the use of the vector identity, equation (Bda5):

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + \nabla \left(\frac{|\underline{u}|^2}{2} \right) - \underline{u} \times (\nabla \times \underline{u}) \right\} = -\nabla p + \underline{f} \tag{Bdb5}$$

In some applications it is convenient to consider Euler's equations in the forms they take in other coordinate systems such as cylindrical and spherical coordinate systems and a separate page is devoted to this.

Moreover, in the case of conservative body forces we may replace $\underline{f}$ by $\nabla\mathcal{U}$ where $\mathcal{U}$ is the body force potential ($\mathcal{U} = -\rho g y$ for the force due to gravity when y is vertically upward) and then equation (Bdb5) can be written as

$$\rho \left\{ \frac{\partial \underline{u}}{\partial t} + \nabla \left(\frac{|\underline{u}|^2}{2} \right) - \underline{u} \times (\nabla \times \underline{u}) \right\} = -\nabla p + \rho\nabla\mathcal{U} \tag{Bdb6}$$

[Note that in a static fluid ($\underline{u} = 0$) under the action of gravity ($\mathcal{U} = -gy$) this equation (Bdb7) reduces to $p = -\rho g y +$ constant consistent with the pressure variation in a fluid at rest.]

If, in addition, the flow is incompressible (ρ is constant and uniform) then this can be written as

$$\frac{\partial \underline{u}}{\partial t} - \underline{u} \times (\nabla \times \underline{u}) + \nabla \left\{ \frac{p}{\rho} - \mathcal{U} + \frac{|\underline{u}|^2}{2} \right\} = 0 \tag{Bdb7}$$

and we will utilize this form in the pages which follow.

Note that in a static fluid ($\underline{u} = 0$) under the action of gravity ($\mathcal{U} = -gy$) this equation (Bdb7) reduces to $\partial p/\partial y = -\rho g$ which is consistent with the expression derived in the section on fluid statics.

Extensive use will be made of Euler's equations during our discussions of fluid flow phenomena. Under some conditions the equations can be integrated to yield a scalar relation between the pressure, velocity and elevation, an important equation known as Bernoulli's equation. However in order to properly frame that derivation and integration it is valuable to digress to discuss a quantity called the vorticity that plays a central role in our understanding of fluid flow.