Circulation

Another important quantity in fluid mechanics that is closely related to the vorticity is the **circulation**. In

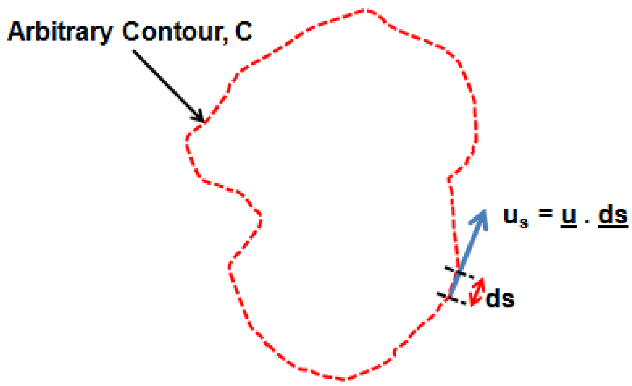

Figure 1: Circulation around an arbitrary closed contour in a flow.

a general three-dimensional flow the circulation, Γ, is defined as the line integral of the tangential velocity around any closed contour, C, in the flow as sketched in Figure 1:

$$\Gamma \;=\; \oint_C u_s ds \;=\; \oint_C \underline{u}.\underline{ds} \tag{Bde1}$$

where the circle on the integral sign indicates a closed contour. While at first sight this may seem a strange quantity we shall find that it plays an important role in many contexts; for example, the circulation around an airfoil is directly related to the lift generated by that foil.

The circulation, Γ, is directly related to the vorticity since, by Stokes' theorem the line integral of equation (Bde1) is equal to

$$\Gamma \;=\; \oint_C \underline{u}.\underline{ds} \;=\; \int_S \nabla \times \underline{u} dS \;=\; \int_S \underline{\omega} dS \tag{Bde2}$$

where S is any surface bounded by the contour C. Consequently the circulation is the integral of the vorticity over any surface bounded by C.

Though it may be superfluous it is instructive to repeat the derivation of the result (Bde2) using a graphical approach. Consider a small element $dx \times dy$ in a planar flow as shown in Figure 2. The tangential velocities on sides AB and DA are denoted by u and $(-v)$ respectively and therefore the tangential velocities on sides BC and CD will be $(v+(\partial v/\partial x)dx)$ and $-(u+(\partial u/\partial y)dy)$ respectively. Consequently the circulation around ABCD is

$$\Gamma \;=\; \oint_{ABCD} \underline{u}.\underline{ds} \;=\; udx + \left\{ v + \frac{\partial v}{\partial x}dx \right\} dy - \left\{ u + \frac{\partial u}{\partial y}dy \right\} dx - vdy \tag{Bde3}$$

After cancellations

$$\Gamma \;=\; \left\{ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\} dxdy = \omega dxdy \tag{Bde4}$$

Then, starting with this as a building block, we can add together the contributions from a whole series of incremental elements to generate the circulation around a much larger contour. For example consider

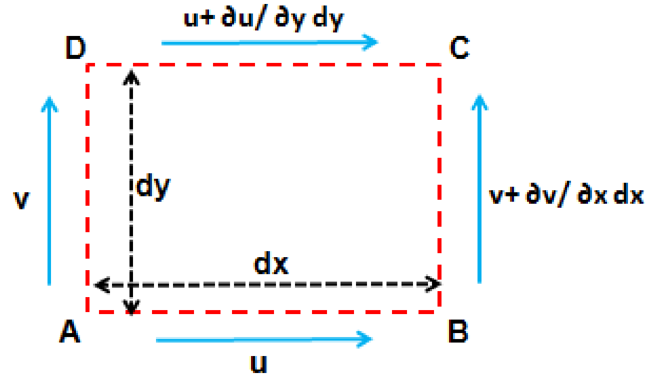

Figure 2: Circulation around an infinitesmal element.

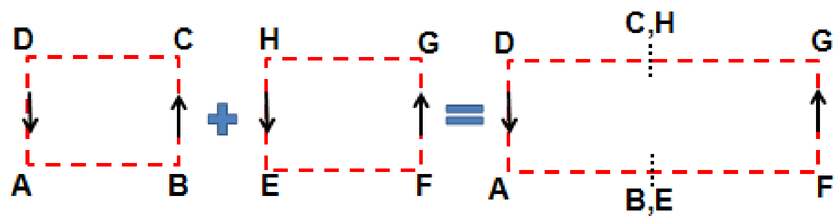

Figure 3: Circulation around two adjacent elements.

the two adjacent incremental elements, ABCD and EFGH, shown in Figure 3. When the ABCD contour integral is combined with the EFGH contour integral the contributions from the adjacent sides BC and HE cancel since the integrals have the opposite sign and therefore the circulation for the contour AFGD is simply given by the sum of the circulations for ABCD and EFGH. It follows that a macroscopic contour integral is simply the sum of its incremental parts or

$$\Gamma = \int_A \omega \, dxdy \tag{Bde5}$$

where A iis the area enclosed by the contour. Though all we have done is to prove a simple version of Stokes' theorem, it is nevertheless useful to visualize the relation between vorticity and circulation in these graphical terms.