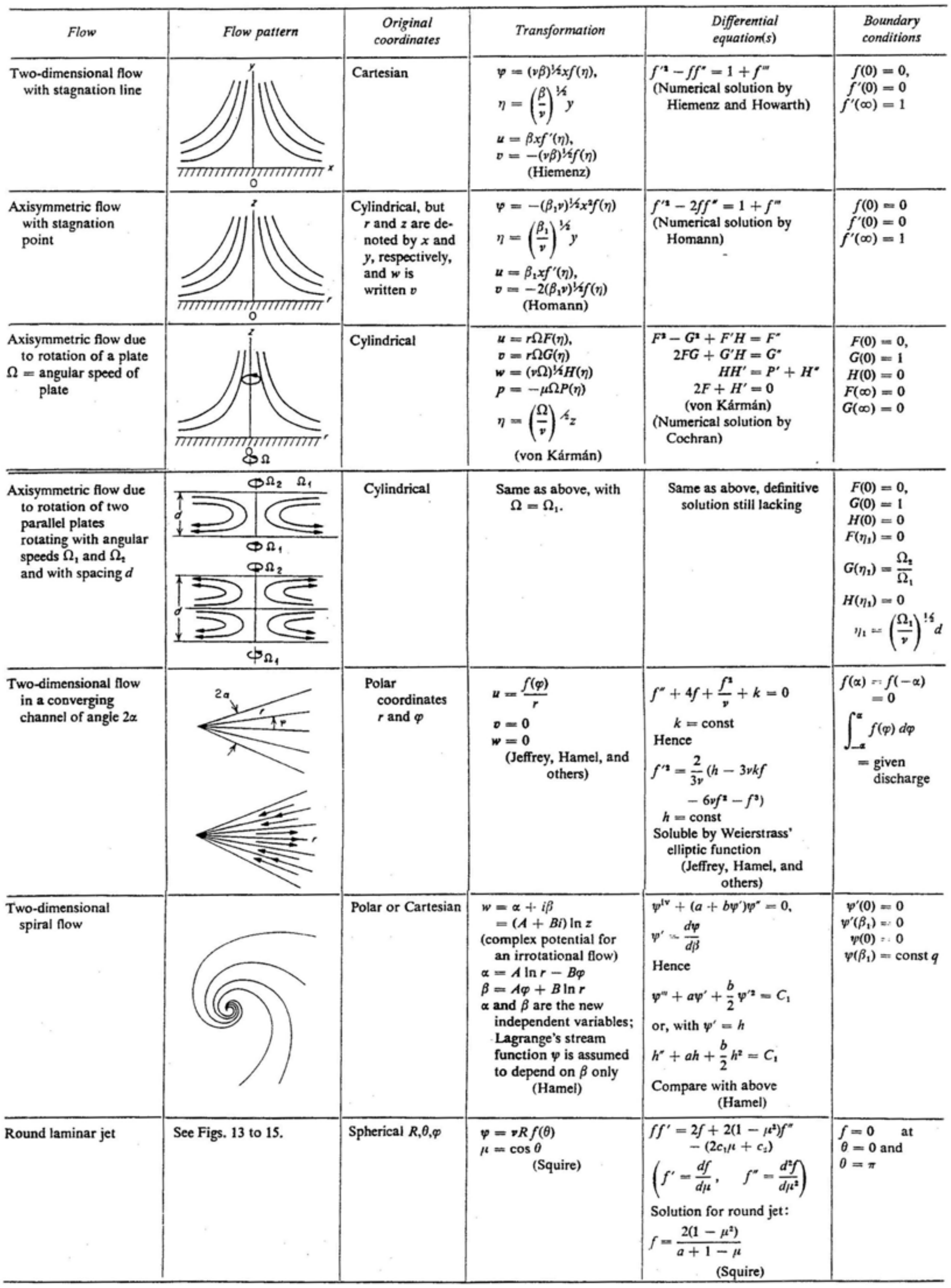
CATEGORY 2 FLOWS

Yih (1969) compiled a list of additional exact solutions to the Navier-Stokes equations for an incompressible



Flow	Flow pattern	Original coordinates	Transformation	Differential equation(s)	Boundary conditions
Two-dimensional flow with stagnation line		Cartesian	$\psi = (\nu\beta)^{1/2} x f(\eta)$, $\eta = \left(\frac{\beta}{\nu}\right)^{1/2} y$ $u = \beta x f'(\eta)$, $v = -(\nu\beta)^{1/2} f(\eta)$ (Hiemenz)	$f'^2 - ff'' = 1 + f'''$ (Numerical solution by Hiemenz and Howarth)	$f(0) = 0$, $f'(0) = 0$ $f'(\infty) = 1$
Axisymmetric flow with stagnation point		Cylindrical, but r and z are denoted by x and y, respectively, and w is written v	$\psi = -(\beta_1\nu)^{1/2} x^2 f(\eta)$ $\eta = \left(\frac{\beta_1}{\nu}\right)^{1/2} y$ $u = \beta_1 x f'(\eta)$, $v = -2(\beta_1\nu)^{1/2} f(\eta)$ (Homann)	$f'^2 - 2ff'' = 1 + f'''$ (Numerical solution by Homann)	$f(0) = 0$ $f'(0) = 0$ $f'(\infty) = 1$
Axisymmetric flow due to rotation of a plate Ω = angular speed of plate		Cylindrical	$u = r\Omega F(\eta)$, $v = r\Omega G(\eta)$ $w = (\nu\Omega)^{1/2} H(\eta)$ $p = -\mu\Omega P(\eta)$ $\eta = \left(\frac{\Omega}{\nu}\right)^{1/2} z$ (von Kármán)	$F^2 - G^2 + F'H = F''$ $2FG + G'H = G''$ $HH' = P' + H''$ $2F + H' = 0$ (von Kármán) (Numerical solution by Cochran)	$F(0) = 0$, $G(0) = 1$ $H(0) = 0$ $F(\infty) = 0$ $G(\infty) = 0$
Axisymmetric flow due to rotation of two parallel plates rotating with angular speeds Ω_1 and Ω_2 and with spacing d		Cylindrical	Same as above, with $\Omega = \Omega_1$.	Same as above, definitive solution still lacking	$F(0) = 0$, $G(0) = 1$ $H(0) = 0$ $F(\eta_1) = 0$ $G(\eta_1) = \frac{\Omega_2}{\Omega_1}$ $H(\eta_1) = 0$ $\eta_1 = \left(\frac{\Omega_1}{\nu}\right)^{1/2} d$
Two-dimensional flow in a converging channel of angle 2α		Polar coordinates r and φ	$u = \frac{f(\varphi)}{r}$ $v = 0$ $w = 0$ (Jeffrey, Hamel, and others)	$f'' + 4f + \frac{f^2}{\nu} + k = 0$ $k = \text{const}$ Hence $f'^2 = \frac{2}{3\nu}(h - 3\nu k f - 6\nu f^2 - f^3)$ $h = \text{const}$ Soluble by Weierstrass' elliptic function (Jeffrey, Hamel, and others)	$f(\alpha) = f(-\alpha) = 0$ $\int_{-\alpha}^{\alpha} f(\varphi)\, d\varphi$ = given discharge
Two-dimensional spiral flow		Polar or Cartesian	$w = \alpha + i\beta = (A + Bi) \ln z$ (complex potential for an irrotational flow) $\alpha = A \ln r - B\varphi$ $\beta = A\varphi + B \ln r$ α and β are the new independent variables; Lagrange's stream function ψ is assumed to depend on β only (Hamel)	$\psi^{iv} + (a + b\psi')\psi'' = 0$, $\psi' = \frac{d\psi}{d\beta}$ Hence $\psi''' + a\psi' + \frac{b}{2}\psi'^2 = C_1$ or, with $\psi' = h$ $h'' + ah + \frac{b}{2}h^2 = C_1$ Compare with above (Hamel)	$\psi'(0) = 0$ $\psi'(\beta_1) = 0$ $\psi(0) = 0$ $\psi(\beta_1) = \text{const}\ q$
Round laminar jet	See Figs. 13 to 15.	Spherical R, θ, φ	$\psi = \nu R f(\theta)$ $\mu = \cos\theta$ (Squire)	$ff' = 2f + 2(1 - \mu^2)f'' - (2c_1\mu + c_2)$ $\left(f' = \frac{df}{d\mu}, \quad f'' = \frac{d^2 f}{d\mu^2}\right)$ Solution for round jet: $f = \frac{2(1 - \mu^2)}{a + 1 - \mu}$ (Squire)	$f = 0$ at $\theta = 0$ and $\theta = \pi$

Figure 1: Table of additional exact solutions to the Navier-Stokes equations. From Yih (1969).

fluid with uniform and constant viscosity. This list is reproduced here as Figure 1. These are all solutions in which it was possible to reduce the partial differential equations to an ordinary differential equation which then required solution by numerical means.