Example: Pump or Turbine

The principle of a rotary pump or turbine provides a good example of the use of the angular momentum theorem. Consider the geometry of and flow through a pump or turbine rotor as depicted in Figure 1.f

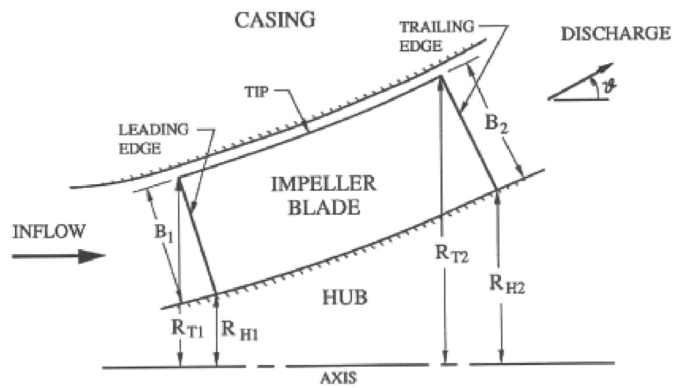

Figure 1: Cross-sectional view through the axis of a pump or turbine rotor.

The flow through a rotor rotating at a speed, Ω (in radians/sec) is visualized by developing a meridional surface (Figure 2) that will be assumed to be an axisymmetric stream surface. On this meridional surface the fluid velocity in a non-rotating coordinate system is denoted by $v(r)$ (with subscripts 1 and 2 denoting particular values at inlet and discharge) and the corresponding velocity relative to the rotating blades is denoted by $w(r)$. The velocities, v and w, have components v_θ and w_θ in the circumferential direction, and v_m and w_m in the meridional direction. Axial and radial components are denoted by the subscripts a and r. As shown in Figure 2, the flow angle $\beta(r)$ is defined as the angle between the relative velocity vector in the meridional plane and a plane perpendicular to the axis of rotation. The blade angle $\beta_b(r)$ is defined as the inclination of the tangent to the blade in the meridional plane and the plane perpendicular to the axis of rotation. If the flow is precisely parallel to the blades, $\beta = \beta_b$. To keep the example simple it is assumed that the blade heights, B_1 and B_2, are small compared with the radii R_{H1} and R_{H2} so that the flows at inlet and discharge can be characterized by a single radius, R_1 and R_2, a single meridional velocity, v_{M1} or v_{M2}, and a single azimuthal velocity, $v_{\theta 1}$ or $v_{\theta 2}$. By continuity of mass it must follow that the mass flow rate, Q, through the machine must be equal to

$$Q \;=\; 2\pi R_1 B_1 v_{M1} \;=\; 2\pi R_2 B_2 v_{M2} \tag{Bei1}$$

We now use the component of the angular momentum theorem in the axial direction in order to relate the torque, T, applied to the fluid within the rotor (and therefore the torque which the shaft applies to the rotor and therefore the torque supplied to the shaft from outside the machine) to the fluid velocities entering and leaving the rotor. The flow in a frame of reference rotating with the rotor is assumed to be steady (the flow in a non-rotating frame is unsteady) and therefore if we choose a control volume which is rotating and contains all the fluid inside the rotor and bounded by axisymmetric surfaces the first term in the angular momentum theorem involving the rate of change of momentum within the control volume will be zero. It remains to determine the net flux of angular momentum out of the control volume. The fluxes of angular momentum entering and leaving the rotor are respectively

$$\rho Q R_1 v_{\theta 1} \quad \text{and} \quad \rho Q R_2 v_{\theta 2} \tag{Bei2}$$

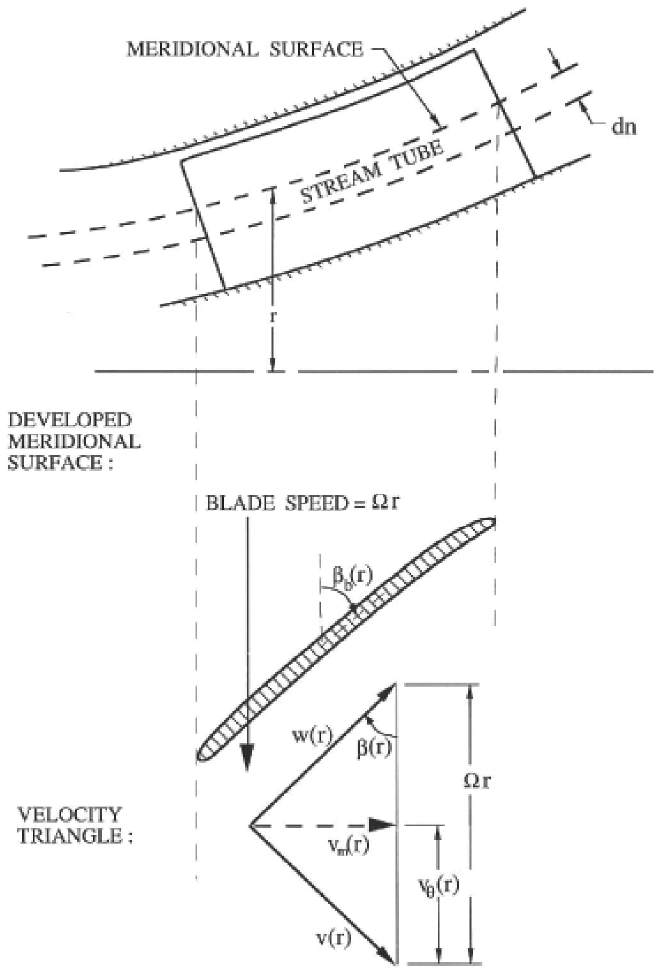

Figure 2: Developed meridional surface and velocity triangle.

and therefore by the angular momentum theorem

$$T = \rho Q(R_2 v_{\theta 2} - R_1 v_{\theta 1}) \tag{Bei3}$$

Moreover, the rate of work done by the rotor on the fluid will be simply $T\Omega$ and, assuming no mechanical losses, this would be equal to the shaft work, $\dot{W}$, entering (or leaving) the machine. Therefore

$$\dot{W} = \rho Q\Omega(R_2 v_{\theta 2} - R_1 v_{\theta 1}) \tag{Bei4}$$

Since we also found when we considered the application of the first law of thermodynamics to a flowing, incompressible fluid that

$$Q(p_2^T - p_1^T) = T\Omega = \dot{W} \tag{Bei5}$$

(where p_2^T and p_1^T are the total pressure at discharge and inlet) we have obtained the following relation between the flow rate, Q and the total pressure change across the machine:

$$p_2^T - p_1^T = \frac{\dot{W}}{Q} = \rho\Omega(R_2 v_{\theta 2} - R_1 v_{\theta 1}) \tag{Bei6}$$

Thus the performance of the machine is known once the azimuthal velocities $v_{\theta 2}$ and $v_{\theta 1}$ have been determined. Often we know the direction of the flow at inlet and therefore can determine $v_{\theta 1}$ from the known velocity v_1 and the known flow angle. Sometimes the flow at discharge is nearly tangential to the blades and therefore $v_{\theta 2}$ can be determined from the inclination of the blades at discharge and the known velocity v_2. The details of departures from these approximations are discussed in the sections on fluid machinery.