Streamfunction

In some particular flows, for example planar flow or axisymmetric flow, we can introduce a special scalar quantity called the stream function which, through its introduction, means that the continuity equation for that flow is automatically satisfied. Normally the stream function is denoted by ψ and we begin by defining the stream function for planar, incompressible flow.

In the section on mass conservation we have seen that the continuity equation for planar, incompressible flow in the xy plane is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{Bch1}$$

and therefore if we define a stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{Bch2}$$

the continuity equation is automatically satisfied. Note that in mathematical terms we have replaced two unknown functions, $u(x,y)$ and $v(x,y)$, with a single unknown function, $\psi(x,y)$, from which the velocities can be derived.

We now explore the physical meaning of this streamfunction by examining how this scalar quantity changes from place to place within the planar flow field. As a result of small incremental displacements, dx and dy, the value of the stream function will change by $d\psi$ where by basic calculus

$$d\psi = \frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=-v\ dx+u\ dy \tag{Bch3}$$

from the definition of ψ. Let us examine first a displacement along a streamline so that, as described in the section on streamlines, $dx = udt$ and $dy = vdt$. Substituting into the above relation it follows that

$$d\psi = -v\ udt+u\ vdt = 0 \tag{Bch4}$$

Therefore the streamfunction is constant along a streamline and we could envision each streamline being labelled with that value.

Now consider how ψ changes with displacements along lines normal to a streamline. To do so we define a coordinate, n, normal to a streamline and an angle, θ, that the streamline makes with the x axis at that point. It follows that a small displacement, dn, in the normal direction (the n direction) will be given by

$$dx = -\ dn\ \sin\theta \quad ; \quad dy = dn\ \cos\theta \tag{Bch5}$$

and if we also denote the magnitude of the velocity at that point by q ($q^2 = u^2 + v^2$) so that $u = q\sin\theta$ and $v = q\cos\theta$ then it follows that the change in the streamfunction, $d\psi$, over a displacement dn normal to the streamline will be

$$d\psi = -\ v\ dx\ +\ u\ dy\ =\ q\ dn \tag{Bch6}$$

In words, the difference between the streamfunction on a streamline and the streamfunction at a point a distance dn from that streamline (in the normal direction) is equal to the volume flow rate (per unit breadth perpendicular to the plane of flow) passing between those two points. Moreover by integrating along a line from one streamline to another we can also say that the difference in the streamfunction values associated with those two streamlines is equal to the volume flow rate (per unit breadth) passing along the streamtube between those two streamlines.