Examples of Streamfunctions for Planar, Incompressible Flows

Having defined the stream function for incompressible planar flow, it is useful to examine the stream functions for some simple flows of this type and to elucidate some of their characteristics:

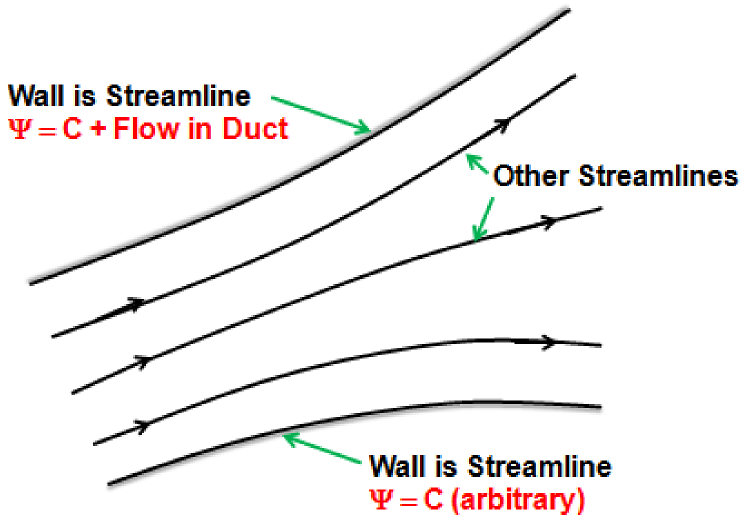

Figure 1: Streamlines in a duct.

- **Flow through a duct**. Since fluid cannot pass through a solid wall or boundary it follows that the velocity vector at that wall must be tangential to that boundary and therefore that the wall must be a streamline. Consequently the streamlines in the flow through a solid-walled duct must be as shown in Figure 1 Since we are free to choose the value of the stream function on one of the streamlines in a flow, it is common to choose one of the walls to be the streamline, $\psi = 0$. It then follows that in the example depicted in Figure 1, the value of the stream function on the other wall will be equal to the volume flow rate through the duct per unit depth normal to the diagram.

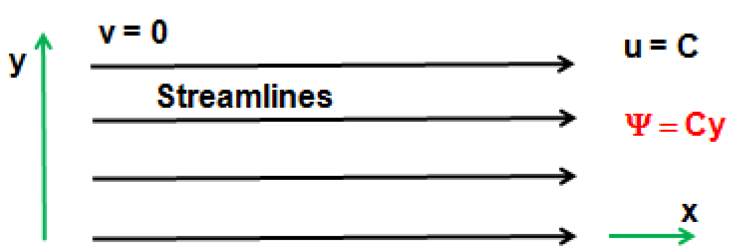

Figure 2: Uniform stream.

- **Uniform Stream**. Perhaps the simplest flow of all is a uniform stream in which the velocity everywhere has the same magnitude and direction. This is called a **Uniform Stream**. If we define a coordinate system such that the x-axis is in the same direction as the uniform stream, then the flow is as depicted in Figure 2 and the stream function and velocity components are

$$\psi = Uy \quad ; \quad u = \frac{\partial \psi}{\partial y} = U \quad ; \quad v = -\frac{\partial \psi}{\partial x} = 0 \tag{Bcj1}$$

where U is the magnitude of the uniform stream velocity. More generally a uniform stream with velocities U and V in the x and y directions would have the stream function $\psi = Uy - Vx$.

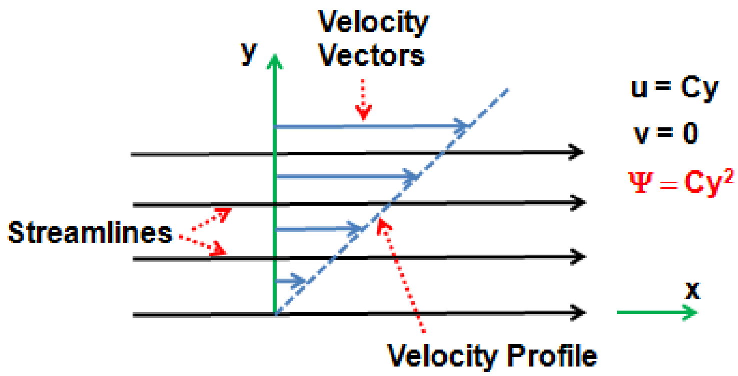

Figure 3: Shear flow in the x direction.

- **Simple Shear Flow**. Since a stream function with linear dependence on the coordinates corresponds to a uniform stream, it is natural to turn next to a stream function with quadratic dependence on the coordinates. First examine the stream function and corresponding velocities given by

$$\psi = \frac{Cy^2}{2} \quad ; \quad u = \frac{\partial \psi}{\partial y} = Cy \quad ; \quad v = -\frac{\partial \psi}{\partial x} = 0 \tag{Bcj2}$$

where C is a uniform constant. This corresponds to a shear flow with velocity vectors in the x direction like a uniform stream but with a velocity, u, that increases linearly in the y direction. This is a **simple shear flow** in the x direction with a "shear rate", C; it is depicted in Figure 3. Clearly a shear flow in an arbitrary direction would be represented by the more general quadratic stream function, $\psi = C_1 y^2 + C_2 x^2$ where C_1 and C_2 are constants.

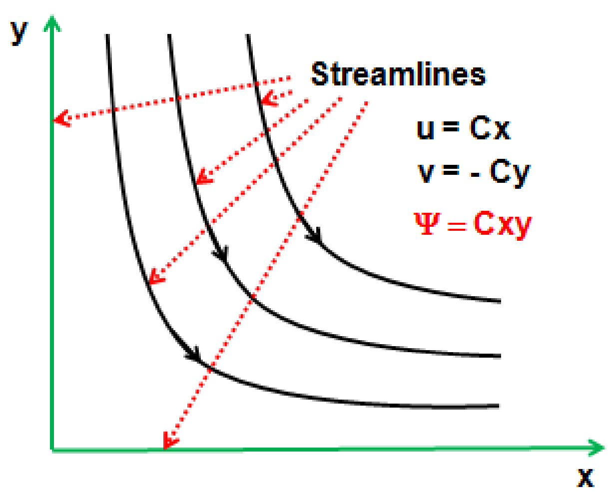

Figure 4: Flow in a rectangular corner.

- **Rectangular Corner Flow**. The other quadratic stream function to explore would be $\psi = Cxy$ for which

$$\psi = Cxy \quad ; \quad u = \frac{\partial \psi}{\partial y} = Cx \quad ; \quad v = -\frac{\partial \psi}{\partial x} = -Cy \tag{Bcj3}$$

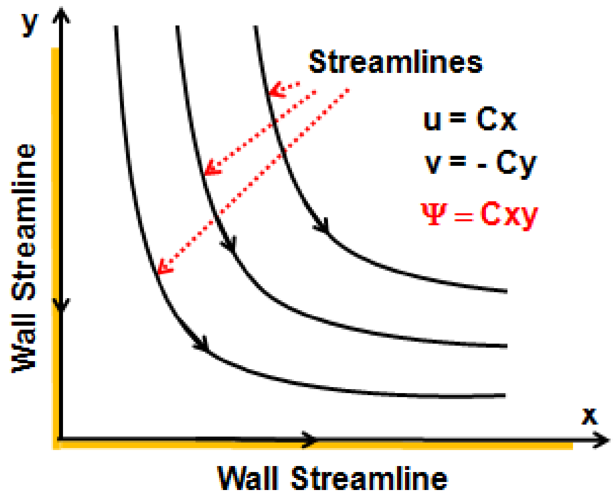

Figure 5: Flow in a rectangular corner with walls.

Clearly this has streamlines that are rectangular hyperbolae (xy = constant) as shown in the first quadrant in Figure 4. Moreover, the x and y axes are also streamlines. Since streamlines and walls are interchangeable in so far as these mass conservation constraints are concerned we could also visualize this as the stream function for the flow in a rectangular corner as depicted in Figure 5. Alternatively if we extend the solution into the other three quadrants we obtain the flow depicted in Figure 6. This is known as an "extensional flow".

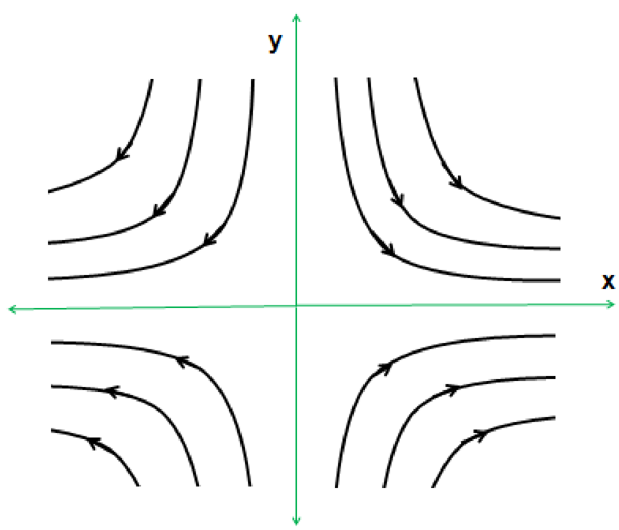

Figure 6: Extensional flow.

- **Polar Coordinates**. Several other simple flows emerge when the equations for the stream function are written in polar coordinates as defined in Figure 7. The components of the velocity in the r and θ directions denoted by u_r and u_θ are then

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad ; \quad u_\theta = -\frac{\partial \psi}{\partial r} \tag{Bcj4}$$

This leads to the following stream functions for some additional simple flows.

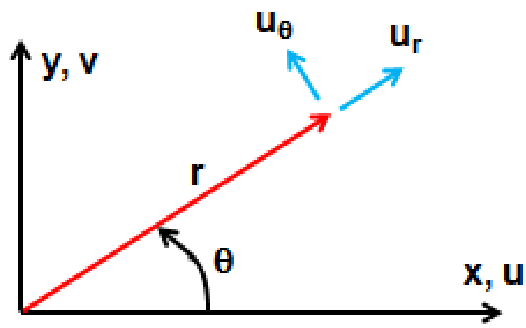

Figure 7: Polar coordinates.

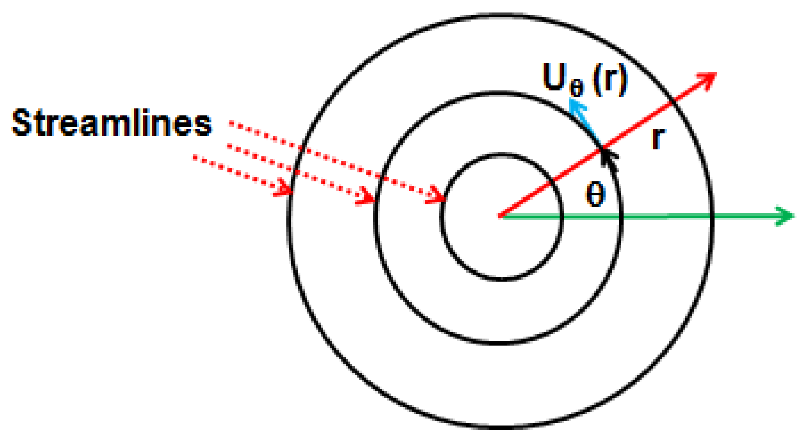

Figure 8: Vortex flow.

- **Vortex Flow**. If the flow in the r direction is zero ($u_r = 0$) then it follows from the above relations that the stream function, $\psi(r)$, is only a function of r and not of θ. The streamlines must therefore be circles as depicted in Figure 8. Flows of this type are known as **vortex flows**. There are several important types of simple vortex flow depending on the precise form of the function, $\psi(r)$. Though we will deal with these in more detail later, we should note that the idealized flow in which

$$\psi(r) = -C \ln r \quad \text{and} \quad u_\theta = \frac{C}{r} \tag{Bcj5}$$

  is known as a **free vortex** and the idealized flow in which

$$\psi(r) = -Cr^2/2 \quad \text{and} \quad u_\theta = Cr \tag{Bcj6}$$

  is known as a **forced vortex**. Note that a free vortex has a velocity which becomes infinite as $r \to 0$ whereas a forced vortex has a velocity which becomes infinite as $r \to \infty$. In the later section which discusses vorticity we shall see that a real vortex typically has a central core in which the velocity behaves like a forced vortex and an outer region in which the velocity behaves like a free vortex, thus avoiding both unrealistic infinities. Viewed from above, a hurricane has a form similar to this.

- **Source and Sink Flow**. In the polar coordinate system, a flow in which the velocity in the θ direction is zero ($u_\theta = 0$) and in which the stream function, $\psi(\theta)$, is only a function of θ (and not of r) clearly represents a source or sink flow.

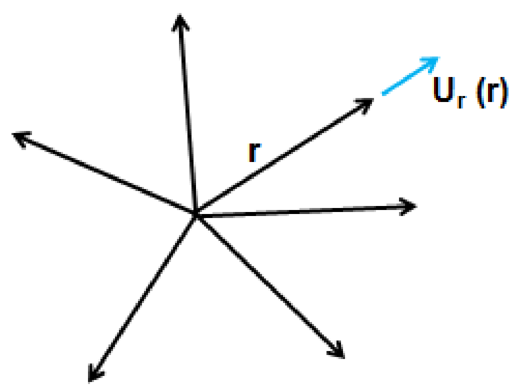

Figure 9: Source or sink flow.