Macroscopic form of continuity

The simplest context in which to apply the principle of conservation of mass is to an internal flow confined within some solid vessel or collection of pipes and in which the flows into and out of that vessel is simply characterized (or approximately characterized). Consider, for example, the solid-walled tank depicted in figure 1 that has three pipes connected to it, labelled A, B and C. For the purpose of the present analyses we choose a control volume that encloses the entire vessel and cuts across the inlet and exit pipes as shown by the red, dashed line in the figure.

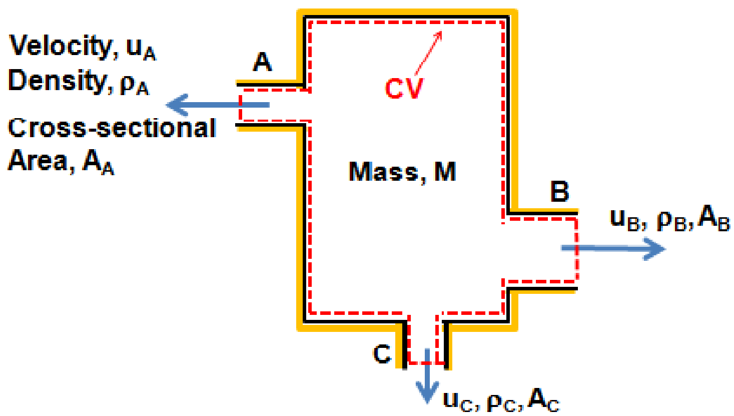

Figure 1: Macroscopic Eulerian control volume.

We denote the mass of the fluid inside the control volume by M, the velocity, density and cross-sectional area of pipe A at the point where the CV cuts across that pipe by u_A, ρ_A and A_A respectively. We similarly characterize the flow through pipes B and C (at the points where the CV cuts across those pipes) by the subscripts B and C. These flow velocities are denoted as positive when directed *out of* the vessel.

Since according to the principle of conservation of mass, fluid mass cannot be created or destroyed, it follows that the rate of flow of mass *into* the control volume must be equal to the rate of increase of mass within the control volume. The latter quantity is readily represented by the time derivative, dM/dt (note that this time derivative is unambiguous since M is a function only of time and not of position). On the other hand the rate of flow of mass *into* the control volume requires a little more construction.

The block of fluid that would cross the surface of the control volume at location A in a time δt has a volume equal to $u_A A_A \delta t$ and therefore the rate of flow of volume (the volume flux) across that part of the control volume surface is $u_A A_A$. Hence the the rate of flow of mass (the mass flux) across that part of the control volume surface is $\rho_A u_A A_A$. Recalling the sign convention for the velocities it follows that the rate at which mass is entering the control volume is given by

$$-\rho_A u_A A_A - \rho_B u_B A_B - \rho_C u_C A_C = -\sum_{a=A}^{all} \rho_a u_a A_a \qquad \text{(Bcc1)}$$

where the extension to the summation over all conduits into and out of the control volume is obvious. In passing we note that we have implicitly assumed that at the location A the velocity, u_A and density, ρ_A

are uniform over the area A_A (and similarly for the other locations). If this is not the case it is clear that the simple forms developed above will have to be replaced by integrals over the areas such as A_A. This is essentially what is done in the integral approach described later.

It follows that, in this example, conservation of mass requires that

$$\frac{dM}{dt} + \rho_A u_A A_A + \rho_B u_B A_B + \rho_C u_C A_C = 0 \tag{Bcc2}$$

or, more generally, that

$$\frac{dM}{dt} + \sum_{a=A}^{all} \rho_a u_a A_a = 0 \tag{Bcc3}$$

and this is a form in which conservation of mass is invoked in a wide range of applications. If the fluid can be considered incompressible then the densities in all of the terms of the above equation are identical and the continuity equation therefore reduces to

$$\frac{dV}{dt} + \sum_{a=A}^{all} u_a A_a = 0 \tag{Bcc4}$$

where V is the volume of the fluid inside the control volume. If the vessel and pipes are rigid this volume V cannot change with time and so the continuity equation is further reduced to

$$\sum_{a=A}^{all} u_a A_a = 0 \tag{Bcc5}$$

This would also be the case if the flow were steady ($dV/dt = 0$) even if the vessel and pipes were not rigid. This particularly simple form of the continuity equation is, perhaps, the most commonly used version in practical applications. It simply states that the volume flow rate in must equal the volume flow rate out. Perhaps the commonest example is the simple steady duct flow shown in figure 2 for which

$$\rho_A u_A A_A = \rho_B u_B A_B \tag{Bcc6}$$

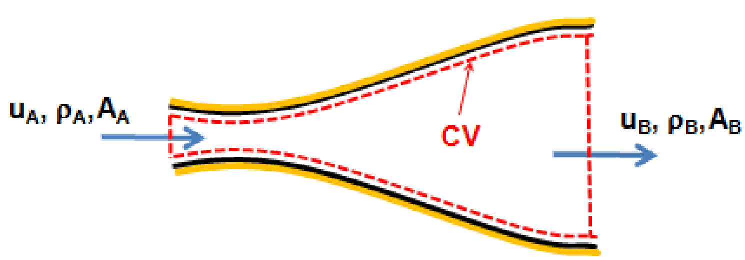

Figure 2: A simple steady duct flow.