Differential form of continuity

In the second or differential approach to the invocation of the conservation of mass, we consider a small Eulerian control volume of fluid within the flow that measures $dx \times dy \times dz$ in some fixed Cartesian coordinate system. Depicted in figure 1, this volume must be small compared with the typical spatial distance within the flow over which substantial changes in the velocities, pressure, etc. vary. However it must also be large compared with the molecular dimensions and mean free paths of the molecules of the fluid, so that it becomes sensible to characterize the fluid motion (and other properties) using continuum quantities. It should be noted that, though such intermediate scales between the global flow scale and the molecular scale can be found in many practical problems, there are flows for which it is not possible to identify such an intermediate scale. In such circumstances one must resort to other methodologies to apply the conservation laws.

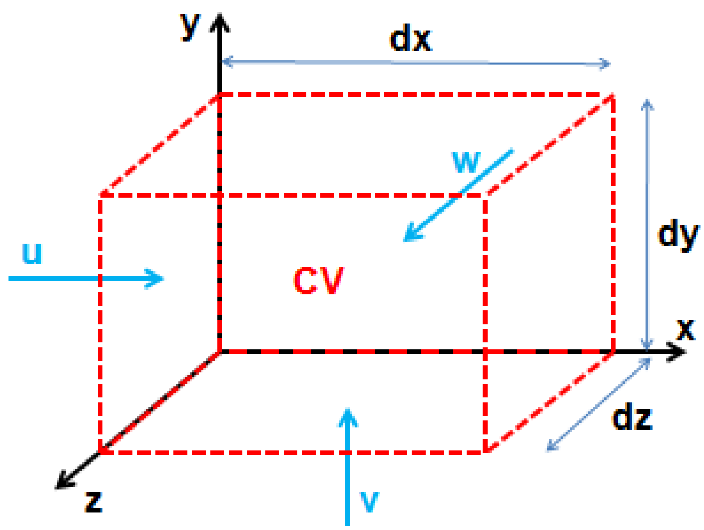

Figure 1: Infinitesmal Eulerian control volume.

Assuming the continuum approximation is valid, we then consider the flux of mass into the differential control volume and equate it with the rate of increase of mass inside the control volume. Consider first the flux of mass through the two sides perpendicular to the $x-$axis. We will define u as the velocity in the $x-$direction at the center of lefthand of these two sides. Similarly we define the density of the fluid at the center of lefthand side as ρ. Then the flux of mass into the control volume through that lefthand side is given by $\rho u dy\ dz$. Then it follows that by Taylor's series (neglecting all terms of order $(dx)^2$ and higher which can be shown to have no contribution to the result) the flux of mass *out of* the control volume through the righthand side is given by $[\rho u + \{\partial(\rho u)/\partial x\}dx]dy\ dz$. Combining the fluxes through these two sides perpendicular to the x-direction, it follows that the net flux of mass *out of* the control volume through the sides perpendicular to the x-direction is $\{\partial(\rho u)/\partial x\}dx\ dy\ dz$. The fluxes through the other two pairs of sides follow from a similar construction so that the net flux of mass *out of* the control volume through all of its sides becomes

$$\left\{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right\} dx\ dy\ dz = \frac{\partial(\rho u_j)}{\partial x_j} dx\ dy\ dz \tag{Bcd1}$$

By conservation of mass this must be equal to minus the rate of increase of mass inside the control volume. Since the mass inside the control volume is $\rho\ dx\ dy\ dz$ the differential form of the continuity equation becomes

$$\left\{ \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right\} dx\ dy\ dz = \frac{\partial(\rho u_j)}{\partial x_j} dx\ dy\ dz = -\frac{\partial(\rho\ dx\ dy\ dz)}{\partial t} \tag{Bcd2}$$

where we must use the Eulerian time derivative since the control volume is defined as an Eulerian volume. Re-arranging and cancelling the differential form of the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{Bcd3}$$

or in tensor notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \tag{Bcd4}$$

or in vector notation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{u}) = 0 \tag{Bcd5}$$

If the flow is steady these clearly reduce to

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \frac{\partial(\rho u_j)}{\partial x_j} = 0 \quad \text{or} \quad \nabla.(\rho \underline{u}) = 0 \tag{Bcd6}$$

If the fluid is incompressible (whether steady or not) they reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u_j}{\partial x_j} = 0 \quad \text{or} \quad \nabla.\underline{u} = 0 \tag{Bcd7}$$

These are the differential forms of the continuity equation in a rectangular Cartesian coordinate system. There are also many problems in which it is much more convenient to use an alternate coordinate system such as a polar coordinate system, a cylindrical coordinate system or a spherical coordinate system. We detail the forms of the continuity equation in these alternate coordinate systems on another page.