Vorticity Transport Equation

For an incompressible Newtonian fluid with a uniform viscosity, the Navier-Stokes equations (Bhf4) are:

$$\rho\frac{Du_i}{Dt} \; = \; \rho\left\{\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right\} \; = \; -\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}+f_i \tag{Bhi1}$$

and for a conservative force field f_i can be written as $\partial\mathcal{U}/\partial x_i$ where $\mathcal{U}$ is the body force potential (in the case of the force due to gravity, $\mathcal{U}=-gy$, where y is the vertically upward coordinate. Alternatively equation (Bhi1) may be written in vector form with $\underline{f}=\nabla\mathcal{U}$ as

$$\rho\left\{\frac{\partial \underline{u}}{\partial t}+\nabla\left(\frac{|\underline{u}|^2}{2}\right)-\underline{u}\times\underline{\omega}\right\} \; = \; -\nabla p+\rho\nabla\mathcal{U}+\mu\nabla^2\underline{u} \tag{Bhi2}$$

where, as usual, the vorticity, $\underline{\omega}=\nabla\times\underline{u}$ and we note the vector identity

$$\nabla^2\underline{u} \; = \; \nabla(\nabla.\underline{u})-\nabla\times\underline{\omega} \tag{Bhi3}$$

Taking the curl of the equation (Bhi2) in order to eliminate the pressure, p, and using two additional vector identities, namely

$$\nabla^2\underline{\omega} \; = \; \nabla(\nabla.\underline{\omega})-\nabla\times(\nabla\times\underline{\omega}) \; = \; -\nabla\times(\nabla\times\underline{\omega}) \tag{Bhi4}$$

since, in the present case, $\nabla.\underline{\omega}$ is zero and

$$\nabla\times(\underline{u}\times\underline{\omega}) \; = \; (\underline{\omega}.\nabla)\underline{u}-\underline{\omega}(\nabla.\underline{u})-(\underline{u}.\nabla)\underline{\omega}+\underline{u}(\nabla.\underline{\omega}) \; = \; (\underline{\omega}.\nabla)\underline{u}-(\underline{u}.\nabla)\underline{\omega} \tag{Bhi5}$$

again since $\nabla.\underline{u}=0$ and $\nabla.\underline{\omega}=0$. Then the curl of equation (Bhi2) can be written as

$$\frac{D\underline{\omega}}{Dt} \; = \; \frac{\partial\underline{\omega}}{\partial t}+(\underline{u}.\nabla)\underline{\omega} \; = \; (\underline{\omega}.\nabla)\underline{u}+\nu\nabla^2\underline{\omega} \tag{Bhi6}$$

or

$$\frac{D\omega_i}{Dt} \; = \; \frac{\partial\omega_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j} \; = \; \omega_j\frac{\partial u_i}{\partial x_j}+\nu\frac{\partial^2\omega_i}{\partial x_j\partial x_j} \tag{Bhi7}$$

This is the *vorticity transport equation* for an incompressible fluid with a uniform and constant viscosity. In using this equation we note that the term $(\underline{\omega}.\nabla)\underline{u}$ is the gradient of the velocity $\underline{u}$ in the direction of the vorticity vector multiplied by the magnitude of $\underline{\omega}$. This term is zero for planar flow since the velocity vector is perpendicular to the vorticity vector.

This is an important result in that it informs us of a number of different characteristics of fluid flow:

[A] In a planar, inviscid flow both of the terms on the right hand side of the vorticity transport equation are zero, the first because the flow is planar and the second because it is inviscid. But in the flow of a uniform stream around a object, the vorticity far upstream of the object is zero because the gradient of the velocity there is zero. Hence the vorticity is zero everywhere in the flow and the flow is *irrotational*.

[B] In a three-dimensional, inviscid flow the vorticity transport equation becomes

$$\frac{D\underline{\omega}}{Dt} \; = \; (\underline{\omega}.\nabla)\underline{u} \tag{Bhi8}$$

Therefore, in a flow that originates with a uniform stream in which the vorticity, $\underline{\omega}$ is zero, the right hand side of equation (Bhi8) is zero and so $D\underline{\omega}/Dt = 0$ and so the vorticity cannot change from zero. Therefore without the effects of viscosity, the vorticity everywhere in such a flow will be zero and the flow will be irrotational.

[C] In a planar viscous flow the vorticity transport equation becomes

$$\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega} \tag{Bhi9}$$

which is a convection/diffusion equation that teaches that vorticity is both convected and diffused in such a flow. Therefore, in the flow that originates with a uniform stream the zero vorticity upstream only changes because vorticity is diffused into the flow by the action of viscosity from some source of vorticity. In many flows that source is the friction with a solid wall where the no-slip condition produces vorticity that diffuses out into the flow. Consider, for example, the flow around an airfoil as

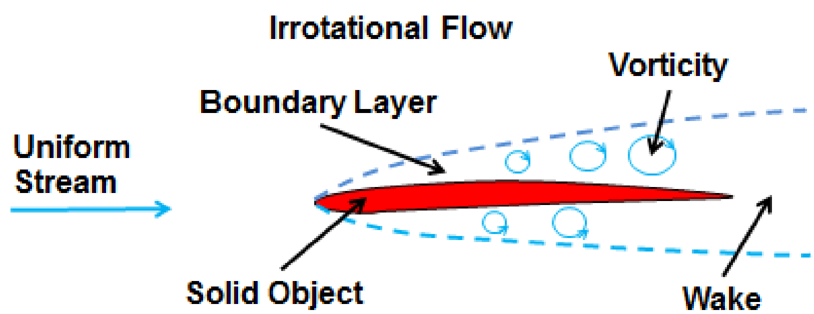

Figure 1: Typical boundary layer in a high Reynolds number flow

depicted in Figure 1. Vorticity is created by the no-slip condition at the solid surface but is also being convected downstream so that (at least at high Reynolds number) it is confined to a thin layer next to the surface beyond which the flow is irrotational. Another example that illustrates this diffusive process is the flow induced by a flat plate bounding fluid and impulsively moved in its own plane to initiate shearing of the fluid. This example is one of the exact solutions to the Navier-Stokes equations detailed in section (Bi).

[D] In a three-dimensional viscous flow the right hand side of the vorticity transport equation (Bhi6) or (Bhi7) teaches that the vorticity in a Lagrangian element of fluid flowing along a streamline will only change for one of two reasons. Either it changes because of the diffusion of vorticity into that element through the action of viscosity. Or it changes because of the term $(\underline{\omega}.\nabla)\underline{u}$, a phenomenon that is known as *vortex stretching*. If we consider a coordinate, s, measured along a line tangent to the vorticity vector (a *vortex line*) and denote the velocity in that direction by u_s then the term $(\underline{\omega}.\nabla)\underline{u}$ is equal to $|\omega|du_s/ds$ where $|\omega|$ denotes the magnitude of the vorticity. Since du_s/ds is the rate at which the fluid element on the vortex line is being stretched in the direction of the vorticity, it is appropriate to consider the term $(\underline{\omega}.\nabla)\underline{u}$ as the contribution of vortex stretching to the rate of change of the vorticity. This stretching is similar to the way the rotation rate of a ballet dancer or ice-skater increases as they draw in their arms; it represents a consequence of the conservation of angular momentum.