The Stress Tensor

The general state of stress in any homogeneous continuum, whether fluid or solid, consists of a stress acting perpendicular to any plane and two orthogonal shear stresses acting tangential to that plane. Thus,

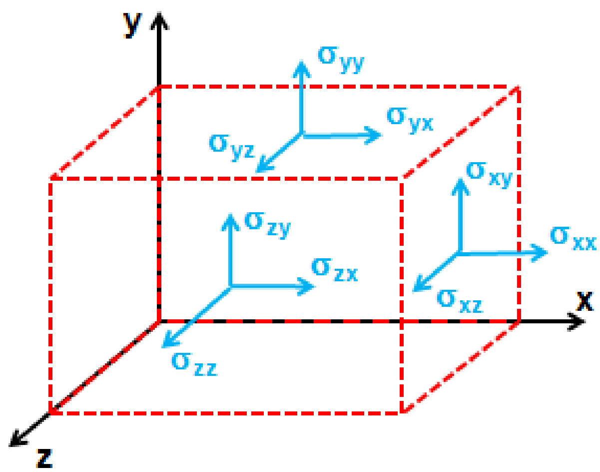

Figure 1: Differential element indicating the nine stresses.

as depicted in Figure 1, there will be a similar set of three stresses acting on each of the three perpendicular planes in a three-dimensional continuum for a total of nine stresses which are most conveniently denoted by the **stress tensor**, σ_{ij}, defined as the stress in the j direction acting on a plane normal to the i direction. Equivalently we can think of σ_{ij} as the 3×3 matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \text{(Bha1)}$$

in which the diagonal terms, σ_{xx}, σ_{yy} and σ_{zz}, are the normal stresses which would be equal to $-p$ in any fluid at rest (note the change in the sign convention in which tensile normal stresses are positive whereas a positive pressure is compressive). The off-digonal terms, σ_{ij} with $i \neq j$, are all shear stresses which would, of course, be zero in a fluid at rest and are proportional to the viscosity in a fluid in motion. In fact, instead of nine independent stresses there are only six because, as we shall see, the stress tensor is symmetric, specifically $\sigma_{ij} = \sigma_{ji}$. In other words the shear stress acting in the i direction on a face perpendicular to the j direction is equal to the shear stress acting in the j direction on a face perpendicular to the i direction. The proof is as follows: consider the very small rectangular element depicted in Figure 2 and evaluate the moment of the forces about the axis parallel to the z direction. The only stresses contributing forces that in turn contribute a moment about an axis parallel to the z-axis are those shown in Figure 2. The two forces on planes perpendicular to the y direction have moment arms of $dy/2$ while the two forces on plane perpendicular to the x direction have moment arms equal to $dx/2$. Therefore, neglecting the higher order contributions that are of order $(dx)^4$, the net moment in the clockwise direction is

$$2\sigma_{yx} dxdz \left\{ \frac{dy}{2} \right\} - 2\sigma_{xy} dydz \left\{ \frac{dx}{2} \right\} \quad \text{(Bha2)}$$

According to Newton's law of angular motion this moment is equal to the moment of inertia multiplied by angular acceleration of the element. The moment of inertia will be of the order of the density multiplied

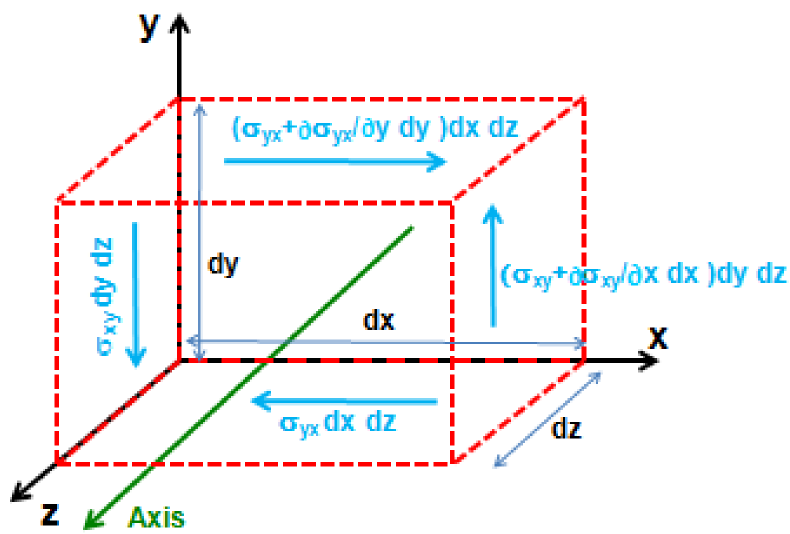

Figure 2: Forces contributing to moment about an axis parallel to z-axis.

by the dimension of the element to the fifth power and since the angular acceleration does not contain a length dimension, it follows that for an element of infinitesmal size the moment given by equation (Bha2) must be zero. It therefore follows that

$$\sigma_{yx} \;=\; \sigma_{xy} \tag{Bha3}$$

The same procedure for moments about axes parallel to the x and the y axes leads to

$$\sigma_{yz} \;=\; \sigma_{zy} \quad \text{and} \quad \sigma_{xz} \;=\; \sigma_{zx} \tag{Bha4}$$

hence proving the symmetry of the stress tensor, $\sigma_{ij} = \sigma_{ji}$.